On Kempton's Generalization of the Negative Binomial Distribution

S.H. Ong[1], Y.C. Low[2*]

[1]Department of Actuarial Science and Applied Statistics, UCSI University, 56000 Kuala Lumpur, Malaysia
[2]Department of Computing and Information Systems, Sunway University, 47500 Selangor, Malaysia
*Corresponding Author: yehchingl@sunway.edu.my, ongsh@ucsiuniversity.edu.my

ABSTRACT   This paper examines the properties of and applications to empirical modelling as well as computation of probabilities of Kempton's generalization of the negative binomial and log-series distributions. Important properties of infinite divisibility and unimodality have been derived. To facilitate computation of the complicated probabilities, practical implementation of the three-term probability recurrence relations is presented. While the generalization of the negative binomial and log-series distributions have been formulated to fit extremely long-tailed count data, the versatility of this generalized negative binomial distribution to fit short-tailed and long-tailed data is illustrated here.

Keywords: infinite divisibility, mixed Poisson, numerical stability, three-term recurrence, unimodality

INTRODUCTION

The statistical modelling and analysis of species abundance is an area of great interest and importance in ecology since the seminal paper by Fisher, Corbet and Williams (1943); see for example, Dallas et al. (2017) and Weber et al. (2017). To model species abundance, Fisher, Corbet and Williams (1943) showed that the number of species s_i, with i individuals in a sample, follows a log-series distribution. This was modelled as follows: Assume that for some species with a given abundance, the number of individuals follows a Poisson distribution, and the abundances among species follow a gamma distribution. The individuals are then seen to follow the negative binomial (NB) distribution. For modelling the species abundance, the NB distribution is truncated at zero since the number of species is unknown. Fisher, Corbet and Williams (1943) noticed that the shape parameter of gamma distribution (mixing distribution) should be allowed to go to 0 while the number of species approaches infinity with their product going to a positive constant c.

Sharing a similar idea, Kempton (1975) considered the generalization of the log-series distribution (GLSD) by deriving the generalization of NB distribution, which is also known as the full beta model (FBM) (see also Holla and Bhattacharya, 1965). Kempton (1975) obtained the FBM as a mixed Poisson distribution with mixing distribution having a probability density function (pdf) given as

$$g(\lambda) = \frac{b^p \lambda^{p-1}}{B(p,q)(1+b\lambda)^{p+q}}. \tag{1.1}$$

Then the resultant distribution has the following probability mass function (pmf)

$$P(k) = \frac{1}{B(p,q)} \int_0^\infty e^{-\lambda} \frac{\lambda^k}{k!} \frac{b^p \lambda^{p-1}}{(1+b\lambda)^{p+q}} d\lambda, \tag{1.2}$$

$$= \frac{\Gamma(p+k)}{k! B(p,q) b^q} \psi(p+q, \ q+1-k; \ 1/b), \qquad k = 0,1,2,...,$$

The form (1.2) is derived from the transformation (Erdélyi et al, 1953, Vol. I, p. 255) of

$$\psi(a,c;x) = x^{1-c} \ \psi(1+a-c, \ 2-c;x).$$

Note that (1.1) is the beta prime distribution with scale parameter $1/b$. It is also known as the inverted beta distribution and is a Pearson Type VI distribution.

The GLSD has pmf

$$P(k) = \frac{\alpha}{k!} \int_0^\infty \frac{x^{k-1} e^{-x}}{(1+bx)^q} dx,$$

$$= \frac{\alpha}{kb^q} \ \psi(q, 1-k+q \ ; 1/b), \qquad k \geq 1$$

where $\alpha^{-1} = \sum_{r=1}^{\infty} \int_0^\infty x^{r-1} e^{-x} (1+bx)^{-q} \ dx / r!$.

Kempton's (1975) interest was in the derivation of the GLSD for modelling species abundance where the frequency count data is characterized by an extremely long tail. However, there was no discussion on the properties of the full beta model. This full beta model (FBM) as a mixed Poisson distribution is of interest on its own because mixed Poisson distributions is an important class of distributions for the statistical analysis of over-dispersed (variance exceeding the mean) count data. A comprehensive survey of mixed Poisson distributions was given by Karlis and Xekalaki (2005), and Gupta and Ong (2005) reviewed the applications of these mixed distributions to very long-tailed count data.

Ong (1995) and Ong and Muthaloo (1995) examined the computations of probabilities by three-term recurrence relations of Kempton's (1975) GLSD and other related distributions (Tripathi, Gupta and White, 1987). These distributions have the interesting feature of being able to fit very long-tailed frequency data. The probability recurrence relations derived by Ong (1995) for the FBM and the GLSD to facilitate computations are respectively

$$k(k+1)P(k+1) = k(k-q-b^{-1})\,P(k) + (k+p-1)\,P(k-1)/b, \qquad (1.3)$$

$$k(k+1)\,P(k+1) = k(k-q-b^{-1})\,P(k) + (k-1)\,P(k-1)/b, \qquad (1.4)$$

for $k \geq 2$. The stability of the recurrence formulae in computation was discussed. A recurrence is said to be stable if the round-off errors do not grow relative to the size of the required function. Recurrence relations (1.3) and (1.4) are stable in the forward direction at a certain value of k onwards (see Ong, 1995, p. 265). In other words, starting the recurrence from this k onwards will give reliable computed probabilities compared to starting from $k = 0$. This feature of the recurrence relations (1.3) and (1.4) makes their implementation in a computer routine difficult since this particular k has to be determined before the recurrence formula can be used. This problem of determining k such that (1.3) and (1.4) can be used safely becomes acute if it is employed in a numerical optimization of the (log) likelihood function. In this situation, a value of k has to be determined for every point (p, q, b) of the parameter space.

The objectives of this paper are to (i) derive important properties of the generalized NB distribution like infinite divisibility and unimodality, (ii) consider the implementation of the recurrence relations (1.3) and (1.4) based upon a heuristic approach of selecting k and (iii) illustrate the versatility of the FBM in fitting not only very long-tailed data (Kempton, 1975, p. 36) but also short-tailed and moderately long-tailed data. Since (1.3) and (1.4) are similar, the discussion would be restricted to the recurrence relation (1.3).

The organization of the paper is as follows. Section 2 gives several probabilistic properties of the FBM. A heuristic method for testing forward stability of three-term recurrence relation for probabilities is presented in Section 3. Section 4 shows the versatility and superiority of the FBM in empirical modelling of count data. Concluding remarks are given in Section 5.

Properties of the Full Beta Model

In this section we give the properties of the FBM. These properties are consequences of the generalized NB as a mixed Poisson distribution. To facilitate the presentation, some properties of the beta prime distribution with pdf (1.1) are first presented.

2.1 Some properties of beta prime distribution

The beta prime distribution has k-th moment given by

$$E\left[X^k\right] = \frac{B(p+k, q-k)}{B(p,q)b^k}, \qquad q \geq k. \qquad (2.1)$$

The moment generating function (mgf) is

$$M_X(t) = E\left[t^X\right] = e^{-t^{1/b}}\, \frac{\Gamma(p+q)}{\Gamma(q)}\, G_{1,2}^{2,0}\!\left(\begin{array}{c} p+q \\ q, 0 \end{array}\middle|\, -t^{1/b}\right) \qquad (2.2)$$

where $G_{\alpha,\beta}^{m,n}$ is the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(\begin{matrix}a_1,...,a_p\\b_1,...,b_q\end{matrix}\middle|x\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=n+1}^{m}\Gamma(a_j+t)\prod_{j=m+1}^{n}\Gamma(1-b_j-t)}x^{-t}dt.$$

The definitions of infinite divisibility and log-concavity are presented next.

Infinitely divisible distributions constitute an important family of distributions in modelling (Steutel, 1983). A distribution function F is said to be infinitely divisible if, for every $n \in N$,

$$X \overset{d}{=} X_{n,1} + X_{n,2} + .\ .\ + X_{n,n}$$

where F is the distribution of X and $X_{n,1}, X_{n,2,}\ .\ .\ , X_{n,n}$ are independent and identically distributed.

A function f is said to be log-concave on the interval I if the function $\log(f)$ is a concave function on I. Log-concave distributions and the various preservation properties of log-concavity have important applications in many areas (Bagnoli and Bergstrom, 2005; Walther, 2009).

The beta prime distribution has the following important properties, summarized in the following theorems.

Theorem 1: The beta prime distribution is infinitely divisible and log-concave.

Proof: For infinitely divisibility see Steutel and van Harn (2003, Appendix B, section 3).

Log-concavity is proven in Borzadaran and Borzadaran (2011, Table 2, p. 209) for the Pearson Type VI distribution with restrictions on the parameters.

Various reliability properties of the beta prime distribution are a consequence of the log-concavity property (see Borzadaran and Borzadaran, 2011, p. 205). In particular, the beta prime distribution is strongly unimodal.

2.2 Properties of the full beta model

Since the FBM is a mixed Poisson distribution, a number of properties can be inferred directly from the mixing distribution, that is, the beta prime distribution. The probability generating function (pgf) of the FBM is obtained from the mgf of the mixing distribution by the formula about mixed Poisson distribution (see, for example, Karlis and Xekalaki, 2005, p.38) and is given as

$$G_X(z) = M_\Lambda(z-1).$$

We get, from (2.2),

$$G_X(z) = e^{-(z-1)^{1/b}} \frac{\Gamma(p+q)}{\Gamma(q)} G_{1,2}^{2,0}\left(\left.\begin{matrix} p+q \\ q,0 \end{matrix}\right| -(z-1)^{1/b}\right). \qquad (2.3)$$

The rth factorial moment $\mu_{(r)}$ of the FBM corresponds to the rth moment of the beta prime distribution (2.1)

$$\mu_{(r)} = \frac{B(p+r, q-r)}{B(p,q)b^r}, \qquad q \geq r.$$

This formula has been derived by Ong and Muthaloo (1995, p. 936) through the evaluation of an integral.

Theorem 2: The FBM is infinitely divisible.

Proof: If the mixing distribution in a mixed Poisson distribution is infinitely divisible, the mixed Poisson distribution is also infinitely divisible (Maceda, 1948).

Furthermore, Theorem 2 connects a mixed Poisson distribution to a compound Poisson distribution.

Theorem 3: The FBM is a compound Poisson distribution with GLSD whose pgf is

$$Q(z) = 1 - \frac{1}{\theta}\{-\log G_X(z)\}, \quad \theta = -\log G_X(0).$$

Proof: A discrete infinitely divisible distribution is a Poisson-stopped sum (compound Poisson) distribution (Feller, 1968). A compound Poisson distribution has pgf of the form (Steutel and van Harn, 2003, Theorem 3.2)

$$G_X(z) = \exp\left(\theta(Q(z) - 1)\right)$$

where $Q(z)$ is the pgf of the summand distribution (cluster distribution). For the FBM, based on (2.3) and equation (3.4) of Steutel and van Harn (2003), $Q(z)$ is given by

$$Q(z) = 1 - \frac{1}{\theta}\{-\log G_X(z)\}, \quad \theta = -\log G_X(0).$$

This is the pgf of Kempton's generalized log-series distribution, GLSD.

Theorem 4: The FBM is unimodal.

Proof: If the mixing distribution in a mixed Poisson distribution is unimodal, the mixed Poisson distribution is also unimodal (Holgate, 1970).

Computation of Probabilities by Recurrence Relation

There are a number of heuristic approaches (Press et al., 2007) to determine the stability of the recurrence relations in the forward direction. One easy-to-implement method is the following: Consider the recurrence

relation which may be started (first) with initial values 1 and 0 and then (second) with 0 and 1 (Press et al., 2007, p. 220). Ten to twenty terms are generated for each of the two starting conditions. If the differences between the corresponding members of the two sequences are of absolute value less than 10, then the recurrence is stable. If the differences increase slowly, the recurrence is mildly unstable but tolerable. Note that in this method of testing stability, the actual starting values of the functions are not required. This approach may be used to determine k such that (1.3) is stable from this k onwards at a particular point (p, q, b) in the parameter space. To illustrate this heuristic approach, a numerical example is given in Table 1 below. The value of k for each column is the starting value of the recurrence formula and the entries in Table 1 are the absolute difference of the two generated sequences. Table 2 reproduced from Ong (1995, Table I) gives the actual values of the probabilities and values computed from different starting points of the recurrence relation (1.3). It is apparent from a comparison of Tables 1 and 2 that the heuristic rule is conservative for this example since at $k = 4$ the recurrence relation (1.3) is already stable.

Table 1. Illustration of heuristic method for recurrence formula (1.3) with $p = 2.42$, $q = 5.02$ $b = 0.072$

k	1	2	4	8	9
0	1	.	.		
1	25.76	1	.		
2	153.11	13.55	.		
3	740.72	59.02	.		
4	2784.96	226.99	1		
5	8657.54	701.62	6.75		
6	22799.10	1850.85	18.61		
7	52018.42	4220.39	50.87		
8	104480.32	8478.76	114.59	1	
9	187175.29	15188.00	231.33	3.03	1
10	302245.26	24526.44	413.49	4.61	2.60
11	443806.09	36012.70	668.44	8.10	3.55
12	596984.88	48443.26	980.90	11.36	5.73
13	740341.88	60075.49	1319.96	15.72	7.28
14	851113.55	69064.70	1636.52	19.14	9.38
15	911427.93	73958.51	1881.74	22.29	10.50
16	.	74088.28	2014.78	23.63	11.48
17	.	.	2018.54	23.87	11.31
18	.	. .	1898.54	22.29	10.80
.			.	.	.
.			.	.	.
21			.	13.23	6.26

The heuristic method discussed above is easy to implement on a computer. This method has been applied over the parameter space (p, q, b) of practical interest for $0 < p,\ q,\ b < 50$ in steps of 0.1. It is found in general that (1.3) is stable at a starting point of $k \leq 10$ when $p, q, b < 25$. If any of the parameters p, q or b is greater than 25, the starting point is $k > 10$. On the basis of this empirical evidence, the following approach may be adopted in computing probabilities of the FBM.

Test the recurrence relation (1.3) for stability at starting points $k = 0, 1, 2, ..., t = 10$. If the starting point is $k \leq 10$, compute $P(k)$ for $k < 10$ by numerical integration or Monte Carlo integration (Ong et al., submitted) and use (1.3) for $k > 10$; otherwise compute all the probabilities by numerical integration or by other methods.

Remark: In fitting short-tailed frequency data, choose t smaller than 1.

Table 2. Forward stability: Recurrence relation (1.3) (Ong, 1995)

k	$P^{(1)}(k)$	$p^{(2)}(k)$	$p^{(3)}(k)$	$p^{(4)}(k)$
0	0.43594736 (-1)**	0.43594736 (-1)*		
1	0.72244871 (-1)	0.72244871 (-1)*	0.72244871 (-1)*	
2	0.85726426 (-1)	0.85721075 (-1)	0.85726426 (-1)*	
3	0.88758952 (-1)	0.88789187 (-1)	0.88759027 (-1)	
4	0.85540095 (-1)	0.85392469 (-1)	0.85539795 (-1)	0.85540095 (-1)*
5	0.79017278 (-1)	0.79571268 (-1)	0.79018457 (-1)	0.79017278 (-1)*
6	0.71070417 (-1)	0.69347405 (-1)	0.71066791 (-1)	0.71070417 (-1)
7	0.62822133 (-1)	0.67358895 (-1)	0.62831693 (-1)	0.62822113 (-1)
8	0.54898205 (-1)	0.45465451 (-1)	0.54876371 (-1)	0.54898205 (-1)
9	0.47613877 (-1)	0.68405032 (-1)	0.47657750 (-1)	0.47613877 (-1)
10	0.41097606 (-1)	0.38501577 (-2)	0.41019025 (-1)	0.41097607 (-1)
11	0.35370453 (-1)	0.95516290 (-1)	0.35497357 (-1)	0.35370453 (-1)
⋮				
37	0.11313668 (-2)	0.11392617 (-2)	0.11313836 (-2)	0.11313668 (-2)
38	0.101706997 (-2)	0.10137907 (-2)	0.10170632 (-2)	0.101706997 (-2)

$p = 2.42$, $\qquad q = 5.02$, $\qquad b = 0.072$

* initial values for recurrence, ** 0.43594736 (-1) = 0.43594736 x 10^{-1} .

$p^{(1)}(k)$ = probabilities computed by numerical integration

$p^{(2)}(k)$ = probabilities computed by (1.4) stating at $k = 0$

$p^{(3)}(k)$ = probabilities computed by (1.4) starting at $k = 1$

$p^{(4)}(k)$ = probabilities computed by (1.4) starting at $k = 4$

Examples of Data Fitting

Kempton (1975, p.36) fitted the FBM to a very long-tailed frequency data set, that is, the species frequency distribution of insect catches. To exemplify the fit of the FBM to very short-tailed and moderately long-tailed frequency data, the following data sets have been considered:

(i) counts of red mites on apple leaves (Garman, 1951),
(ii) claims in automobile insurance (Bhati et al., 2015),
(iii) hospital stays (Gomez-Deniz and Calderın-Ojeda, 2015), and
(iv) number of sickness absences for men (Taylor, 1967).

The automobile insurance claims data is short-tailed whereas the data on number of sickness absences for men is a moderately long-tailed frequency data.

The parameters are estimated by maximum likelihood estimation via numerical maximization of the likelihood function. The results of the fit as judged by the chi-square values are presented in Tables 3 to 6. For the counts of red mites, the FBM fit as well as the non-central negative binomial (NNB) distribution (Ong and Lee, 1979), which is another extension of the negative binomial distribution. Table 4 displayed the fits for the number of claims in automobile insurance and compared the FBM with the generalized Poisson-Lindley (GPL) distribution of Bhati et al. (2015). The FBM fit is clearly better. For the hospital stays data (Table 5) fitted by Gomez-Deniz and Calderın-Ojeda (2015) to a generalized geometric distribution, the FBM again fits much better. The FBM is compared to the Poisson-inverse Gaussian distribution (Sichel, 1971) for Taylor's data in Table 6; there is an overall improvement in fit.

Table 3. Counts of red mites on apple leaves ($\hat{p} = 1.180, \hat{q} = 38.729, \hat{b} = 0.041$)

No. of mites per leaf	Leaves observed	Negative binomial	NNB $(\lambda = 2)$ (Ong and Lee, 1979)	FBM
0	70	64.49	69.98	69.94
1	38	39.03	35.66	36.93
2	17	20.96	20.75	19.68
3	10	10.97	11.45	10.59
4	9	5.66	6.06	5.75
5	3	2.90	3.11	3.14
6	2	1.48	1.55	1.74
7	1	0.75	0.76	0.97
≥ 8	0	0.76	0.68	1.27
Total	150	150.00	150.00	150.00
χ^2		2.93	2.45	2.45
p-value		0.9387	0.9640	0.9640

Table 4. Number of claims in automobile insurance ($\hat{p}=8.98, \hat{q}=3.38, \hat{b}=19.40$)

Number of claims	Claims Observed	NB	GPL (Bhati et al., 2015)	FBM
0	1563	1566.4	1564.50	1563.94
1	271	261.5	264.25	267.54
2	32	40.146	39.69	36.36
3	7	5.99	5.59	5.57
4	2	0.88	0.87	1.11
Total	1875			
χ^2		3.60	3.49	1.64
p-value		0.4628	0.4794	0.8016

Table 5. Hospital Stays Data $(\hat{p}=0.54, \hat{q}=3.91, \hat{b}=0.63)$

Number of days	Days Observed	NB	Generalized Geometric (Gomez-Deniz and Calderín-Ojeda, 2015)	FBM
0	3541	3544.63	3539.81	3539.73
1	599	583.35	606.12	605.96
2	176	177.5	161.2	165.08
3	48	62.12	56.53	55.04
4	20	23.35	23.06	21.18
5	12	8.81	10.2	9.14
6	5	3.52	4.72	4.34
7	1	1.32	2.24	2.23
8	4	0.44	1.08	1.22
Total	4406			
χ^2		34.73	12.05	6.89
p-value		0.00003	0.1490	0.5485

Table 6. Number of sickness absences ($\hat{p}=2.20, \hat{q}=6.14, \hat{b}=0.05$)

Number of absences	Observed Frequency	Poisson-inverse Gaussian (Sichel, 1971)	FBM
0	31	20.1	28.03
1	35	43.5	44.71
2	55	57.3	52.32
3	59	60.7	54.03
4	49	57.7	52.22
5	59	51.9	48.51
6	41	45.3	43.94
7	38	38.9	39.12
8	32	33.3	34.43
9	31	28.4	30.07
10	24	24.2	26.11
11	22	20.7	22.58
12	22	17.7	19.49
13	17	15.2	16.80
14	16	13.1	14.47
15	10	11.4	12.46

16	8	9.8	10.73
17	8	8.6	9.25
18	11	7.5	7.98
19	4	6.5	6.90
20	3	5.7	5.97
21	6	5.0	5.17
22 – 23	9	8.3	8.39
24 – 25	4	6.4	6.36
26 – 27	7	5.1	4.85
28 – 30	6	5.7	5.26
31 – 35	8	6.1	5.26
36 – 48	8	8.9	7.58
Total	623	623.00	623.00
χ^2		19.81	15.33
p-value		0.8386	0.9645

CONCLUDING REMARKS

Some important probabilistic properties, like infinite divisibility, have been derived for Kempton's full beta model. Although the probability mass function is complicated, its three-term recurrence relation facilitates computation and therefore its applications in statistical analysis. However, numerical stability of the recurrence formula is an issue. A heuristic method for testing forward stability of a three-term recurrence relation for probabilities have been presented. This heuristic rule allows the determination of the point where the recurrence relation should be started; it is easy to implement on the computer and does not require actual starting points of the recurrence formula. The implementation for the GLSD is similar. Four examples of data fitting for the FBM are also given to demonstrate its versatility and superiority to fit both short and moderately long-tailed data. The statistical literature is full of extensions and generalizations of basic count models like the Poisson and negative binomial distributions. It is shown that an existing model, the full beta model, can be used as a good empirical model.

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