THE BALAKRISHNAN-ALPHA-SKEW-NORMAL DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Received: 3rd Jul 2019 Accepted: 4th Mar 2020 Published: 30th Jun 2020
DOI: https://doi.org/10.22452/mjs.vol39no2.5

ABSTRACT In this paper a new type of alpha skew distribution is proposed under Balakrishnan (2002) mechanism and some of its related distributions are investigated. The moments and distributional properties and some extensions related to this distribution are also studied. Suitability of the proposed distribution is tested by conducting data fitting experiments and model adequacy is checked via AIC and BIC in comparison with some related distributions. Likelihood ratio test is carried out to discriminate between normal and proposed distribution.

Keywords: Skew Normal Distribution, Alpha Skew Normal Distribution, Balakrishnan Skew Normal Distribution, Bimodal Distribution.

Math Classification: 60E05, 62E10

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1. INTRODUCTION

One cannot undermine the applications and the value of normal distribution in real life to model the symmetric data. Now there are many real life situations which seem to be symmetric but due to influences of other factors they depart from symmetry (for details see Chakraborty and Hazarika, 2011, and Chakraborty et al., 2015). To tackle these situations Azzalini (1985) discovered the skew-normal distribution by inducting an additional parameter to introduce asymmetry in the normal distribution and is define as follows: A continuous random variable (r.v.) Z follows skew normal (SN) distribution i.e. $Z \sim \mathrm{SN}(\lambda)$ if it has probability density function (pdf) given by

$$f_Z(z;\lambda) = 2\phi(z)\Phi(\lambda z); \quad z, \lambda \in R \tag{1}$$

where, ϕ and Φ are respectively, the pdf and cumulative distribution function (cdf) of the standard normal distribution. Balakrishnan in 2002 as a discussant in Arnold and Beaver (2002) proposed the generalization of the skew normal density and studied its properties. The pdf of the same distribution is

$$f_Z(z;\lambda,n) = \phi(z)[\Phi(\lambda z)]^n / C_n(\lambda); \quad z, \lambda \in R \tag{2}$$

where, n is a positive integer and $C_n(\lambda) = \mathrm{E}(\Phi^n(\lambda U))$, $U \sim \mathrm{N}(0,1)$. In particular, if $\lambda = 1$ the Balakrishnan skew normal density becomes skew normal density of Azzalini (1985). Furthermore, Sharafi and Behbodian (2008) extensively studied its different forms and properties. Bahrami et al. (2009) introduced the two parameter Balakrishnan skew normal distribution. Yadegari et al. (2008) discussed the generalization of Balakrishnan skew normal distribution.

In 2007, Huang and Chen proposed the method for the construction of skew-symmetric distributions starting from a symmetric (about 0) pdf $h(.)$ by establishing the concept of skew function $G(.)$ which is a Lebesgue measurable function such that, $0 \le G(z) \le 1$ and $G(z) + G(-z) = 1,\ z \in R$, almost everywhere. An r.v. Z is said to be skew symmetric if its pdf is of the form:

$$f(z) = 2h(z)G(z);\ z \in R \tag{3}$$

Olivero in 2010 developed a new form of skew distribution which exhibits both unimodal as well as bimodal behavior and named it as alpha skew normal distribution with the pdf given by:

$$f(z;\alpha) = \{(1-\alpha z)^2 + 1\}\varphi(z)\big/(2+\alpha^2);\ z, \alpha \in R \tag{4}$$

Using the same approach of Olivero (2010), Harandi and Alamatsaz (2013) and Hazarika and Chakraborty (2014) explored the alpha skew Laplace distribution and alpha skew Logistic distribution respectively. Venegas et al. (2016) and Louzada et al. (2017) studied the logarithmic form and bivariate form of alpha-skew-normal distribution, respectively. Sharafi et al. (2017) discussed the generalization of alpha-skew-normal distribution.

In this article, the main aim is to propose a new version of alpha skew normal distribution (known as Balakrishnan alpha skew normal $BASN_2(\alpha)$ distribution, where $\alpha \in R$) which is flexible enough to adequately support both uni-modal and bi-modal behaviors as well as positive and negative skewness by considering the methodology advocated by Balakrishnan in 2002 and some of its basic properties are investigated. To exhibit the applicability of the proposed distribution, the two real life datasets are consider which give better fitting when compared to some other known distributions.

The article is organized as follows. In section 2, the Balakrishnan alpha skew normal distribution is defined and some of its important distributional properties are discussed. The half Balakrishnan-alpha-skew normal distribution is defined in section 3. Section 4 discusses about the extensions of Balakrishnan alpha skew normal distribution along with some of its basic properties. The location-scale extension, method of moments and maximum likelihood estimation are given in section 5. In section 6, some numerical examples based on real life data are provided. Finally, the article ended with conclusions in section 7.

2. BALAKRISHNAN ALPHA SKEW NORMAL DISTRIBUTION

In this section we introduce the generalized version of bimodal skew normal distribution of Olivero (2010) and proposed Balakrishnan alpha skew normal distribution.

Definition 1: A r.v. Z is said to follow generalized bimodal normal distribution if it has the following pdf

$$f(z) = \frac{z^n}{C}\varphi(z); \quad z \in R \tag{5}$$

where, n is positive even integers and C is normalizing constant. Symbolically, we can write $Z \sim \mathrm{BN}(n)$. The shapes of pdfs with different choices of n are shown in Figure 1.

Remark 1: The pdf in eqn. (5) has at most two modes and the same has been seen from the equation $f'(z) = \frac{z^{n-1}(z^2 - n)}{C}\varphi(z) = 0$. This equation has only three zero, therefore the pdf in eqn. (5) have only two modes.

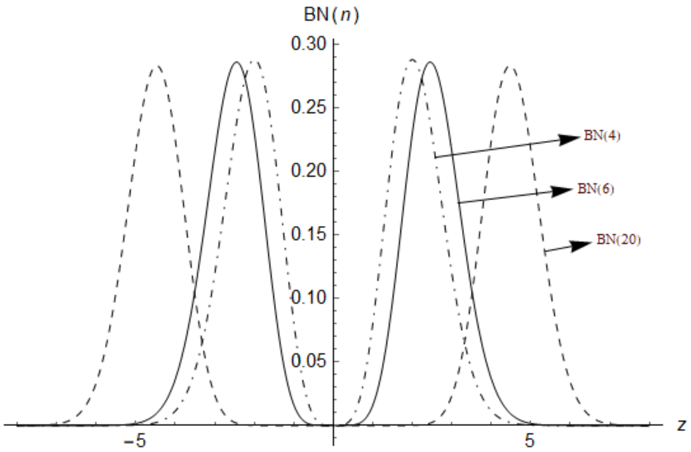
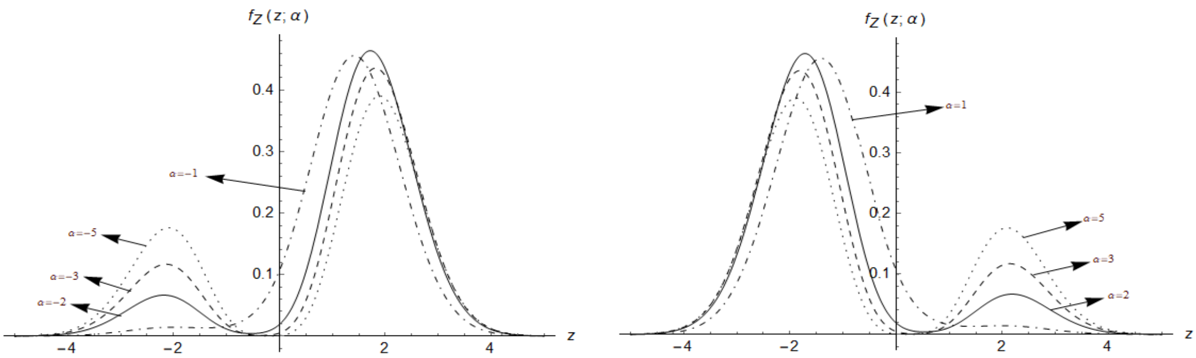


Figure 1. Plots of pdf of $\mathrm{BN}(n)$

Definition 2: An r.v. Z with the pdf given by

$$f_Z(z;\alpha) = \frac{1}{C_2(\alpha)}\left(\frac{(1-\alpha z)^2 + 1}{2+\alpha^2}\right)^2 \varphi(z); \; z, \alpha \in R \tag{6}$$

where, $C_2(\alpha) = 3 - \frac{4}{2+\alpha^2}$, is said to follow **Balakrishnan alpha skew normal** distribution with parameter α. We denote it by $BASN_2(\alpha)$. The plots of the pdfs are depicted in Figure 2 for different choices of the parameter α.



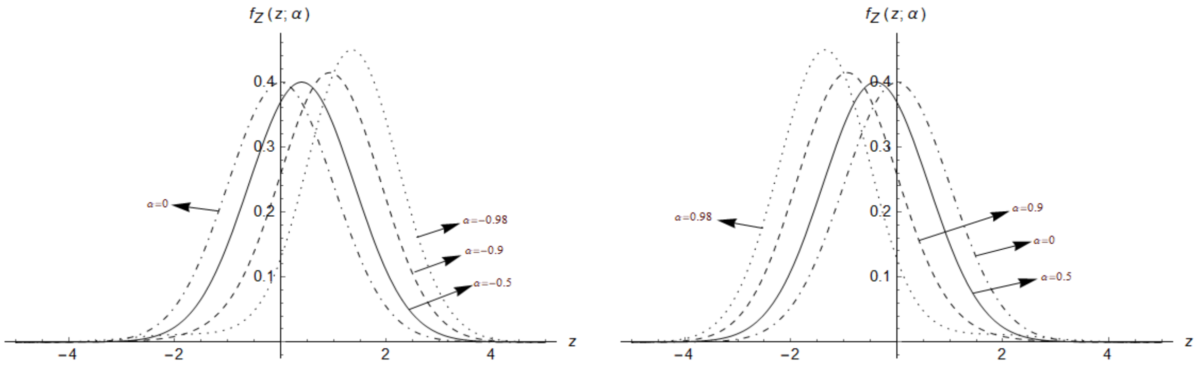


Figure 2. Plots of pdf of $BASN_2(\alpha)$

It's obvious to note and check in Figure 2 that $BASN_2(\alpha)$ is bimodal when $|\alpha| \geq 1$.

Remark 2: The pdf of the proposed $BASN_2(\alpha)$ distribution is constructed using the formula (2), by taking $\Phi(.) = \frac{(1-\alpha z)^2 + 1}{2+\alpha^2}$ and $n = 2$.

Properties of $BASN_2(\alpha)$:

i) $BASN_2(0) = \varphi(z)$

ii) If $\alpha \to \pm\infty$, then pdf of Z becomes $f_Z(z) = \frac{z^4}{3}\varphi(z)$ i.e., $Z \sim BN(4)$

iii) If $Z \sim BASN_2(\alpha)$, then $-Z \sim BASN_2(-\alpha)$

iv) $BASN_2(\alpha)$ has at most two modes.

Proof: To show $BASN_2(\alpha)$ distribution have at most two modes, which is equivalent to prove that the following equation have three zeros.

$$Df_Z(z;\alpha) = \frac{[(1-\alpha z)^2 + 1](\alpha^2 z^3 - 4\alpha^2 z - 2\alpha z^2 + 4\alpha + 2z)\varphi(z)}{C_2(\alpha)(2+\alpha^2)^2} = 0 \qquad (7)$$

It is easy to show that the eqn. (7) has at most three real zeros because $(1-\alpha z)^2 + 1 = 0$ will have two complex roots, $\alpha^2 z^3 - 4\alpha^2 z - 2\alpha z^2 + 4\alpha + 2z = 0$ has three real roots and $\varphi(z) \neq 0$. The same can be depicted from the contour plot of the eqn. (7) given in Figure 3. It is also observed from the Figure 3 that approximately for $-0.95 < \alpha < 0.95$; $BASN_2(\alpha)$ remains unimodal.

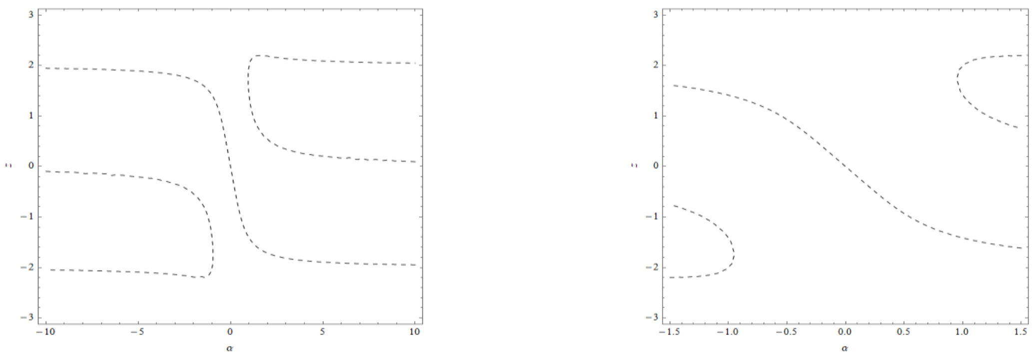


Figure 3. The contour plots of the equation $Df_Z(z;\alpha) = 0$

Proposition 1: If $Z \sim BASN_2(\alpha)$ distribution then its cdf is given by

$$F_Z(z;\alpha) = \Phi(z) + \frac{\alpha\left\{8 - 8\alpha z + 4\alpha^2(2+z^2) - \alpha^3 z(3+z^2)\right\}}{C_2(\alpha)(2+\alpha^2)^2}\varphi(z) \tag{8}$$

where, $\Phi(z)$ is the cdf of standard normal distribution.
Proof: see Appendix A.

The plots of cdf with different choices parameter α is shown in Figure 4.

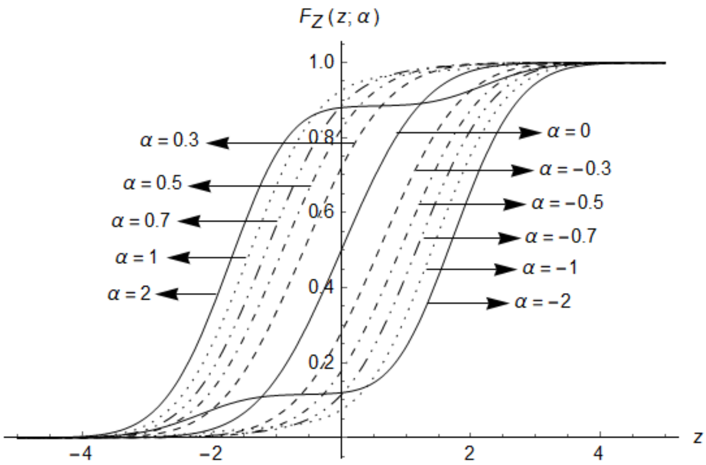


Figure 4. Plots of cdf of $BASN_2(\alpha)$

For $-1 < \alpha < 0 \, (0 < \alpha < 1)$, we can say that the standard normal is stochastically smaller (larger) than $BASN_2(\alpha)$ as seen in the Figure 4.

Remark 3: In particular, if $\alpha \to \pm\infty$ then the cdf of $BASN_2(\alpha)$ becomes the cdf of $BN(4)$ and is given by $F_Z(z) = \Phi(z) - \frac{z(3+z^2)}{3}\varphi(z)$.

Proposition 2: If $Z \sim BASN_2(\alpha)$ distribution then

$$E(Z^n) = \begin{cases} \dfrac{2^{\frac{-(n+4)}{2}}\left\{\dfrac{\alpha^4(n+4)!}{((n+4)/2)!} + \dfrac{16\alpha^2(n+2)!}{((n+2)/2)!} + \dfrac{16n!}{(n/2)!}\right\}}{C_2(\alpha)(2+\alpha^2)^2}, & \text{when } n \text{ is even} \\ \dfrac{-2^{\frac{1-n}{2}}\alpha\left\{\dfrac{\alpha^2(n+3)!}{((n+3)/2)!} + \dfrac{4(n+1)!}{((n+1)/2)!}\right\}}{C_2(\alpha)(2+\alpha^2)^2}, & \text{when } n \text{ is odd} \end{cases} \tag{9}$$

Proof: see Appendix B.

Remark 4: The expression (9) can be rewritten with the help of Gamma function as

$$E(Z^n)=\begin{cases}\dfrac{2^{\frac{n}{2}}\{4+(1+n)\alpha^2(8+(3+n)\alpha^2)\}\,\Gamma(\frac{1+n}{2})}{C_2(\alpha)\,\Gamma(1/2)(2+\alpha^2)^2}, & \text{when } n \text{ is even}\\[2ex] -\dfrac{2^{\frac{5+n}{2}}\alpha\{2+(2+n)\alpha^2\}\Gamma(1+\frac{n}{2})}{C_2(\alpha)\Gamma(1/2)(2+\alpha^2)^2}, & \text{when } n \text{ is odd}\end{cases}$$

In particular, $E(Z)=\dfrac{-4\alpha}{(2+\alpha^2)}$, $E(Z^2)=5-\dfrac{4}{(2+\alpha^2)}-\dfrac{4}{(2+3\alpha^2)}$, $Var(Z)=\dfrac{(2+5\alpha^2)(4+3\alpha^4)}{(2+\alpha^2)^2(2+3\alpha^2)}$ $E(Z^3)=\dfrac{-12\alpha(2+5\alpha^2)}{4+8\alpha^2+3\alpha^4}$ and $E(Z^4)=35-\dfrac{48}{(2+\alpha^2)}-\dfrac{16}{(2+3\alpha^2)}$.

Remark 5: By optimizing $E(Z)$ and $Var(Z)$ with respect to α we get the following bounds.

i. $-1.414\le E(Z)\le 1.414$

ii. $0.972\le Var(Z)\le 4.7966$

The same can be easily visualized from Figure 5 and Figure 6.

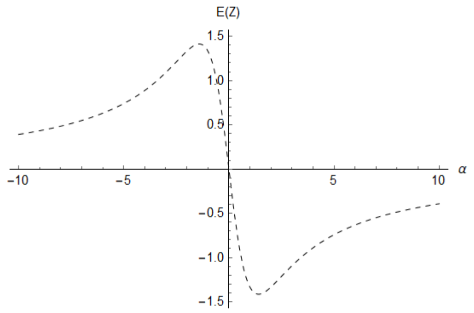


Figure 5. Plot of mean

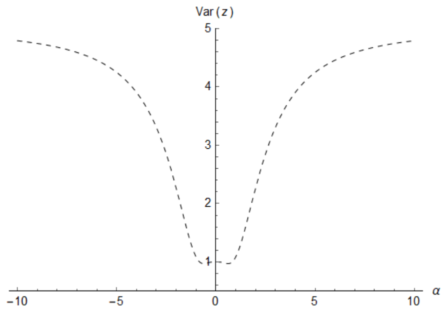


Figure 6. Plot of variance

Remark 6: The expression for skewness (β_1) and kurtosis (β_2) are respectively given by

$$\beta_1=\frac{64\alpha^6(2+3\alpha^2)(4+15\alpha^4)^2}{(8+20\alpha^2+6\alpha^4+15\alpha^6)^3}\text{ and}$$

$$\beta_2=\frac{3(2+3\alpha^2)(32+112\alpha^2+144\alpha^4+216\alpha^6+410\alpha^8+35\alpha^{10})}{(8+20\alpha^2+6\alpha^4+15\alpha^6)^2}$$

By optimizing β_1 and β_2 with respect to α we get the following bounds.

i. $2.5359\le\beta_1\le 0$

ii. $6.7684\le\beta_2\le 3$

The same can be easily visualized from Figure 7 and Figure 8.

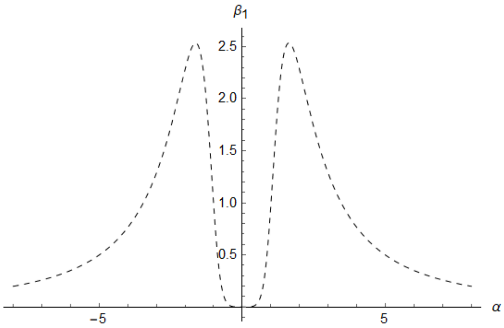


Figure 7. Plot of skewness

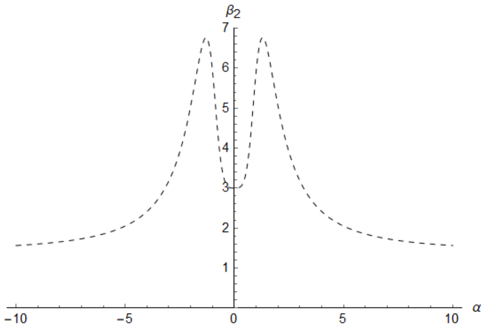


Figure 8. Plot of kurtosis

Proposition 3: If $Z \sim BASN_2(\alpha)$ distribution then its moment generating function (mgf) is given by

$$M_Z(t) = \frac{M_X(t)[\alpha^4 t^4 + 6\alpha^4 t^2 + 3\alpha^4 - 4\alpha^3 t^3 - 34\alpha^3 t + 8\alpha^2 t^2 + 8\alpha^2 - 8\alpha t + 4]}{C_2(\alpha)\ (2+\alpha^2)^2} \quad (10)$$

where, $M_X(t)$ is the mgf of standard normal variable.
Proof: see Appendix C.

Proposition 4: The $BASN_2(\alpha)$ distribution can be represented as a mixture of two components as given below

$$f(z;\alpha) == \frac{\alpha^4 z^4 + 8\alpha^2 z^2 + 4}{C_2(\alpha)\ (2+\alpha^2)^2}\varphi(z) + \frac{(-4\alpha^3 z^3 - 8\alpha z)}{C_2(\alpha)\ (2+\alpha^2)^2}\varphi(z) \quad (11)$$

where the 1st part is a symmetric pdf denoted by $SCBASN_2(\alpha)$ with cdf and mgf given respectively by

$$F(z) = \Phi(z) - \frac{\alpha^2[\alpha^2 z^3 + 3z\alpha^2 + 8z]}{C_2(\alpha)(2+\alpha^2)^2}\varphi(z) \quad (12)$$

$$M_Z(t) = \frac{[\alpha^4 t^4 + 6\alpha^4 t^2 + 3\alpha^4 + 8\alpha^2 t^2 + 8\alpha^2 + 4]}{C_2(\alpha)(2+\alpha^2)^2} M_X(t) \quad (13)$$

where, $\Phi(z)$ is the cdf of standard normal distribution.
Proof: see Appendix D.

Remark 7: (i) For $\alpha = 0$, $SCBASN_2(\alpha)$ becomes standard normal distribution. (ii) $SCBASN_2(\alpha)$ can be useful in generating random numbers from $BASN_2(\alpha)$ stated in the next remark.

Remark 8: To generate the random number Z from $BASN_2(\alpha)$ distribution for different choices of the parameter α one can adopt the acceptance sampling method with the following steps:

I: Generate random number U from Uniform (0,1)

II: Generate random number H from $SCBASN_2(\alpha)$.

III: Set $Z = H$ if $U < \frac{1}{\Delta}\frac{f(H)}{f_1(H)}$, otherwise, step back to I and continue the process.

Where, $\Delta = Sup\left[\frac{f(Z)}{f_1(Z)}\right] = \frac{1}{3}(3+2\sqrt{2})$ and $f(.)$ and $f_1(.)$ are pdf of $BASN_2(\alpha)$ and $SCBASN_2(\alpha)$ respectively.

3. HALF $BASN_2(\alpha)$ DISTRIBUTION

A half Balakrishnan-alpha-skew normal $HBASN_2(\alpha)$ distribution truncated below '0' is given by

$$f_T(t;\alpha) = \frac{[(1-\alpha t)^2+1]^2}{3\alpha^4 - 8\alpha^3 b + 8\alpha^2 - 8\alpha b + 4}\psi(t); \quad t > 0 \tag{14}$$

where, $\psi(t)$ is the pdf of the standard half-normal distribution and $b = \sqrt{\frac{2}{\pi}}$.

This can be considered as a potential life time distribution. The corresponding survival function $S_T(t;\alpha)$ and the hazard rate functions $h_T(t;\alpha)$ of $HBASN_2(\alpha)$ can be expressed as below

$$S_T(t;\alpha) = \frac{\psi(t)\,\alpha(8 - 8\alpha t + 8\alpha^2 + 4\alpha^2 t^2 - 3\alpha^3 t - \alpha^3 t^3) - C_2(\alpha)(2+\alpha^2)^2\,\overline{\Psi}(t)}{-4 + \alpha[8b + \alpha(-8 + 8\alpha b - 3\alpha^2)]}$$

and

$$h_T(t;\alpha) = \frac{[(1-\alpha t)^2+1]^2}{\alpha(-8 + 8\alpha t - 8\alpha^2 - 4\alpha^2 t^2 + 3\alpha^3 t + \alpha^3 t^3) + C_2(\alpha)(2+\alpha^2)^2\psi(t)\overline{\Psi}(t)}.$$

where, $\Psi(t)$ and $\overline{\Psi}(t)$ are respectively the cdf and survival function of the standard half-normal distribution.

We have plotted the $h_T(t;\alpha)$ for the suitable values of the parameter α, in Figure 9 to study its behavior graphically.

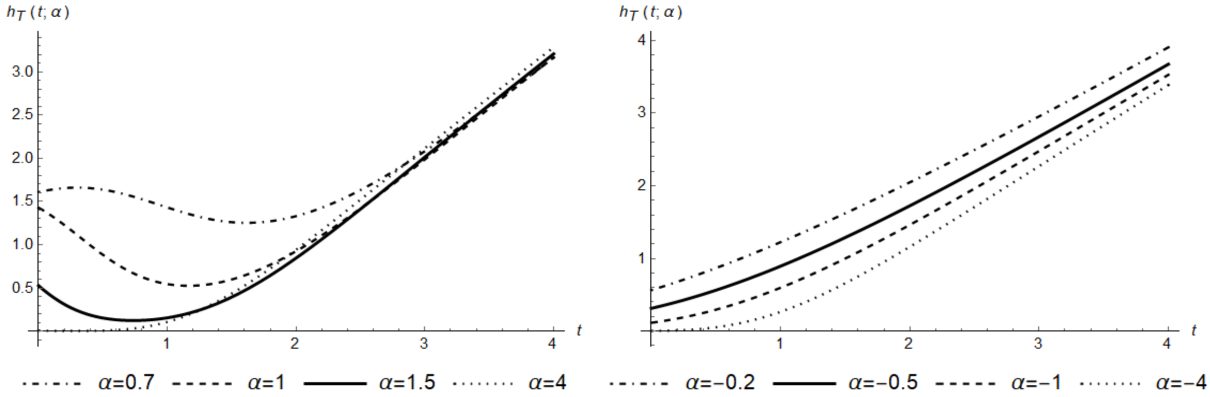


Figure 9. Plots of hazard rate function of $HBASN_2(\alpha)$

It can be observed from Figure 9 that the hazard rate is increasing for $\alpha \le 0$ while it assumes bathtub shape for $\alpha > 0.7$. For the values of $0 < \alpha \le 0.7$, the hazard rate takes different shapes. Therefore, the hazard rate function of $HBASN_2(\alpha)$ distribution assumes different useful shapes depending on the choice of the values of the parameter α, and thus has the potential to be a flexible life time model.

Remark 9: In particular for $\alpha = 0$, $HBASN_2(\alpha)$ distribution reduces to standard half-normal distribution.

4. SOME EXTENSIONS OF $BASN_2(\alpha)$ DISTRIBUTION

In this section we briefly discussed some of the possible extensions of $BASN_2(\alpha)$ distributions. These extensions are being currently investigated and will be reported later.

4.1. The Bivariate $BASN_2(\alpha)$ Distribution

Definition 3: A random vector $\boldsymbol{Z} = (Z_1, Z_2)$ has two dimensional (bivariate) $BASN_2(\alpha)$ distribution if it has the following pdf

$$f(\mathbf{z};\alpha_1,\alpha_2,\rho) = \frac{[(1-\alpha_1\mathbf{z}_1-\alpha_2\mathbf{z}_2)^2+1]^2}{C(\alpha_1,\alpha_2,\rho)}\varphi_2(\mathbf{z}); \mathbf{z}\in R^2,\ \alpha_1,\alpha_2\in R \tag{15}$$

where, $C(\alpha_1,\alpha_2,\rho) = (2+\alpha_1{}^2+2\rho\alpha_1\alpha_2+\alpha_2{}^2)(2+3\alpha_1{}^2+6\rho\alpha_1\alpha_2+3\alpha_2{}^2)$, and $\varphi_2(\boldsymbol{z})$ is the pdf of a bivariate normal distribution $N_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$. We denote it by $\boldsymbol{Z} \sim BBASN_2(\alpha_1,\alpha_2,\rho)$

Special cases of $BBASN_2(\alpha_1,\alpha_2,\rho)$:

- If $\alpha_1 = \alpha_2 = 0$, then $\boldsymbol{Z} \sim N_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right) = \varphi_2(\mathbf{z})$.

- If $\alpha_2 = 0$, then pdf of Z_1 is $\frac{[(1-\alpha_1 z_1)^2+1]^2}{4+8\alpha_1^2+3\alpha_1^4}\varphi_2(z_1)$ and if $\alpha_1 = 0$, then pdf of Z_2 is $\frac{[(1-\alpha_2 z_2)^2+1]^2}{4+8\alpha_2^2+3\alpha_2^4}\varphi_2(z_2)$.
- If $\alpha_1 = \alpha_2 \to \pm\infty$, then $f(\boldsymbol{z};\alpha_1,\alpha_2,\rho) = \frac{(z_1+z_2)^4}{12(1+\rho)^2}\varphi_2(\boldsymbol{z}); \quad \boldsymbol{z} \in R$.
- If $\boldsymbol{Z} \sim BBASN_2(\alpha_1,\alpha_2,\rho)$, then $-\boldsymbol{Z} \sim BBASN_2(-\alpha_1,-\alpha_2,\rho)$.

4.2 A Two- parameter $BASN_2(\alpha)$ Distribution

Definition 4: An r.v. Z has a two-parameter $BASN_2(\alpha)$ distribution with parameters $\alpha_1, \alpha_2 \in R$, denoted by $TPBASN_2(\alpha_1,\alpha_2)$, if its pdf is

$$f(z;\alpha_1,\alpha_2) = \frac{\Phi^2(\alpha_1 z)\Phi^2(\alpha_2 z)}{C(\alpha_1,\alpha_2)}\varphi(z); \qquad z \in R \tag{16}$$

where,

$$C(\alpha_1,\alpha_2) = 32\alpha_1\alpha_2(2+3\alpha_2^2) + 48\alpha_1^3\alpha_2(2+5\alpha_2^2) + 4(4+8\alpha_2^2+3\alpha_2^4) + 8\alpha_1^2(4+24\alpha_2^2 + 15\alpha_2^4) + 3\alpha_1^4(4+40\alpha_2^2+35\alpha_2^4);$$

$\Phi^2(\alpha_1 z) = [(1-\alpha_1 z)^2+1]^2$; and $\Phi^2(\alpha_2 z) = [(1-\alpha_2 z)^2+1]^2$.

Special cases of $TPBASN_2(\alpha_1,\alpha_2)$:

- If $\alpha_1 = \alpha_2 = 0$, then $Z \sim N(0,1) = \varphi(z)$.
- If $\alpha_2 = 0$, then $Z \sim BASN_2(\alpha_1)$ and if $\alpha_1 = 0$, then $Z \sim BASN_2(\alpha_2)$.
- If $\alpha_1 = \alpha_2 = \alpha$, then $f(z;\alpha) = \frac{((1-\alpha z)^2+1)^4}{16+128\alpha^2+408\alpha^4+480\alpha^6+105\alpha^8}\varphi(z) = BASN_4(\alpha)$.
- If $\alpha_1 = \alpha_2 \to \pm\infty$, then $f(z) = \frac{z^8}{105}\varphi(z)$.
- If $Z \sim TPBASN_2(\alpha_1,\alpha_2)$, then $-Z \sim TPBASN_2(-\alpha_1,-\alpha_2)$.

4.3 Balakrishnan Alpha-Beta Skew Normal Distribution

Definition 5: If the pdf of an r.v. Z is given by

$$f(z;\alpha,\beta) = \frac{[(1-\alpha z-\beta z^3)^2+1]^2}{C(\alpha,\beta)}\varphi(z); \quad z,\alpha,\beta \in R \tag{17}$$

then we say that Z is distributed according to the Balakrishnan alpha-beta skew normal distribution with parameters α and β.

where, $C(\alpha,\beta) = 4+3\alpha^4+60\alpha^3\beta+12\alpha\beta(4+315\beta^2)+\alpha^2(8+630\beta^2)+15\beta^2(8+693\beta^2)$. We denote it by $Z \sim BABSN_2(\alpha)$.

Special cases of $BABSN_2(\alpha)$:

- If $\beta = 0$, then we get $BASN_2(\alpha)$ distribution and is given by $f(z) = \frac{[(1-\alpha z)^2 + 1]^2}{C_2(\alpha)(2+\alpha^2)^2}\varphi(z)$.
- If $\alpha = 0$, then we get $f(z) = \frac{[(1-\beta z^3)^2 + 1]^2}{(4+15\beta^2(8+693\beta^2))}\varphi(z)$.
  This equation is known as Balakrishnan beta skew normal ($BBSN_2(\alpha)$) distribution.
- If $\alpha = \beta = 0$, then we get the standard normal distribution.
- If $\alpha \to \pm\infty$, then we get the bimodal normal ($BN(4)$) distribution given by $f(z) = \frac{z^4}{3}\varphi(z)$.
- If $\beta \to \pm\infty$, then we get the bimodal normal ($BN(12)$) distribution given by $f(z) = \frac{z^{12}}{10395}\varphi(z)$.
- If $Z \sim BABSN_2(\alpha,\beta)$, then $-Z \sim LBABSN_2(-\alpha,-\beta)$.

4.4 Generalization of $BASN_2(\alpha)$ Distribution

Definition 6: If the pdf of an r.v. Z is given by

$$f(z;\alpha,\lambda) = \frac{[(1-\alpha z)^2 + 1]^2}{C(\alpha,\lambda)}\varphi(z)\Phi(\lambda z); \quad z,\alpha \in R, \lambda > 0, \tag{18}$$

then we say that Z is distributed according to the Generalized $BASN_2(\alpha)$ distribution with parameters α and λ.

Where, $C(\alpha,\lambda) = (2+4\alpha^2+1.5\alpha^4) - b(2\alpha^3(1-\delta^2)\delta + 4\alpha\delta + 4\alpha^3\delta)$; $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$; $b = \sqrt{\frac{2}{\pi}}$;

$\varphi(z)$ *and* $\Phi(\lambda z)$ are defined above. We denote it by $Z \sim GBASN_2(\alpha,\lambda)$.

Special cases of $GBASN_2(\alpha,\lambda)$:

- If $\alpha = 0$, then $Z \sim SN(\lambda)$.
- If $\lambda = 0$, then $Z \sim BASN_2(\alpha)$.
- If $\alpha = \lambda = 0$, then $Z \sim N(0,1)$.
- If $\alpha \to \pm\infty$, then $f(z;\alpha,\lambda) \to \frac{2z^4}{3}\varphi(z)\Phi(\lambda z)$.
- If $\lambda \to +\infty$, then $f(z;\alpha,\lambda) \to \frac{[(1-\alpha z)^2 + 1]^2}{(2+4\alpha^2+1.5\alpha^4) - b(2\alpha^3+4\alpha+4\alpha^3)}\varphi(z)I(z>0)$,

  and if $\lambda \to -\infty$, then $f(z;\alpha,\lambda) \to \frac{[(1-\alpha z)^2 + 1]^2}{(2+4\alpha^2+1.5\alpha^4) - b(2\alpha^3+4\alpha+4\alpha^3)}\varphi(z)I(z<0)$,

  where, $b = \sqrt{\frac{2}{\pi}}$ and $I(.)$ is an indicator function.
- If $Z \sim GBASN_2(\alpha,\lambda)$, then $-Z \sim GBASN_2(-\alpha,-\lambda)$.

4.6 The Log- $BASN_2(\alpha)$ Distribution

In this section, using the work of (Venegas et al., 2016), we present the definition and some simple properties of log- $BASN_2(\alpha)$ distribution.

Let $Z = e^Y$, then $Y = Log(Z)$, therefore, the density function of Z is defined as follows:

Definition 7: If the pdf of an r.v. Z is given by

$$f(z;\alpha) = \frac{[(1-\alpha\, y)^2 + 1]^2}{C_2(\alpha)\,(2+\alpha^2)^2}\,\varphi(y); \quad z > 0, \alpha \in R \tag{19}$$

then we say that Z is distributed according to the log- $BASN_2(\alpha)$ distribution with parameter α . Where, $y = Log(z)$ and $\varphi(y)$ is the pdf of the standard log-normal distribution. We denote it by $Z \sim LBASN_2(\alpha)$.

Special cases of $LBASN_2(\alpha)$:

- If $\alpha = 0$, then we get the standard log-normal distribution given by $f(z) = \frac{\varphi(y)}{z}$.
- If $\alpha \to \pm\infty$, then we get the log-bimodal normal $LBN(4)$ distribution given by $f_Z(z) = \frac{y^4}{3z}\varphi(y)$.
- If $Z \sim LBASN_2(\alpha)$, then $-Z \sim LBASN_2(-\alpha)$.

5. PARAMETER ESTIMATION OF $BASN_2(\alpha)$

If $Z \sim BASN_2(\alpha)$ distribution then $Y = \mu + \sigma Z$ is the location (μ) and scale (σ) extension of Z and has the pdf is given by

$$f_Y(y;\mu,\sigma,\alpha) = \frac{1}{C_2(\alpha)}\left(\frac{\left\{1-\alpha\left(\frac{y-\mu}{\sigma}\right)\right\}^2 + 1}{2+\alpha^2}\right)^2 \frac{e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}; \ (y,\mu,\alpha) \in \mathrm{R} \text{ and } \sigma > 0 \tag{20}$$

Symbolically, we write as $Y \sim BASN_2(\alpha,\mu,\sigma)$.

5.1. Method of Moments

Let $Y_1, Y_2, ..., Y_n$ be a random sample of siz n drawn from $BASN_2(\alpha,\mu,\sigma)$ distribution in eqn. (20) and m_1, m_2 and m_3 are the first three sample raw moments respectively. Then, the moment estimates of the three parameters μ, σ and α are obtained by

$$m_1 = \mu - \frac{4\alpha\sigma}{2+\alpha^2} \Rightarrow \mu = \frac{4\alpha\sigma}{2+\alpha^2} + m_1 \tag{21}$$

$$m_2 = \mu^2 - \frac{\sigma(16\alpha\mu + 24\alpha^3\mu) - \sigma^2(4\sigma^2 + 24\alpha^2\sigma^2 + 15\alpha^4)}{(2+\alpha^2)(2+3\alpha^2)} \tag{22}$$

$$m_3 = \mu^3 + \frac{-3\sigma(8\alpha\mu^2 + 12\alpha^3\mu^2 - 4\mu\sigma - 24\alpha^2\mu\sigma - 15\alpha^4\mu\sigma + 8\alpha\sigma^2 + 20\alpha^3\sigma^2)}{4 + 8\alpha^2 + 3\alpha^4} \tag{23}$$

Substituting the value of μ from eqn. (21) in eqn. (22) and solving for σ^2, we get

$$\sigma^2 = \frac{(m_2 - m_1^2)(2+\alpha^2)^2(2+3\alpha^2)}{(8 + 20\alpha^2 + 6\alpha^4 + 15\alpha^6)} \tag{24}$$

Finally, by putting these values of μ and σ^2 in eqn.(23), we get the following equation in α

$$m_3 = \frac{\begin{bmatrix} [C_2(\alpha)\, d_1^{\,2}(d_3 d_4 m_1 - 4\alpha d_1 d_2(m_1^{\,2} - m_2))^3 + 12\alpha\, d_1^{\,2}\, d_2^{\,2}(d_3 d_4 m_1 - 4\alpha d_1 d_2(m_1^{\,2} - m_2))^2(m_1^{\,2} - m_2) + \\ 3d_1^{\,4}\, d_2^{\,2} d_5(d_3 d_4 m_1 - 4\alpha d_1 d_2(m_1^{\,2} - m_2))(m_1^{\,2} - m_2)^2 + 12\alpha d_1^{\,6}\, d_2^{\,3} d_3(m_1^{\,2} - m_2)^3] \end{bmatrix}}{C_2(\alpha)\, d_1^{\,2} d_3^{\,3} d_4^{\,3}} \tag{25}$$

where,

$d_1 = 2 + \alpha^2$; $d_2 = 2 + 3\alpha^2$; $d_3 = 2 + 5\alpha^2$; $d_4 = 4 + 3\alpha^4$; and $d_5 = 4 + 24\alpha^2 + 15\alpha^4$.

Furthermore, the value of α is estimated numerically as the exact solution of the eqn.(25) is not easily tractable. Once α is estimated, the rest of the two parameters namely, μ and σ can be estimated directly from eqn. (21) and eqn. (24) respectively.

5.2. Maximum Likelihood Method

Likelihood function:

Let $Y_1, Y_2, ..., Y_n$ be a random sample of size *n* drawn from $BASN_2(\alpha, \mu, \sigma)$ distribution of eqn. (20), then the log-likelihood function for $\theta = (\alpha, \mu, \sigma)$ is given by

$$l(\theta) = 2\sum_{i=1}^{n} \log\left[\left\{1 - \alpha\left(\frac{y_i - \mu}{\sigma}\right)\right\}^2 + 1\right] - n\log(2+\alpha^2) - n\log\sigma - n\log(2+3\alpha^2) - \frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i - \mu}{\sigma}\right)^2 \tag{26}$$

Differentiating this eqn. (26) above partially with respect to the parameters $\alpha, \mu, and\ \sigma$, the following likelihood equations are obtained:

$$\frac{\partial l(\theta)}{\partial \mu} = -\sum_{i=1}^{n} - \frac{(y_i - \mu)}{\sigma^2} + 2\sum_{i=1}^{n} \frac{2\alpha b_i}{\sigma(1 + b_i^{\,2})}$$

$$\frac{\partial l(\theta)}{\partial \sigma} = -\frac{n}{\sigma} - \sum_{i=1}^{n} -\frac{(y_i - \mu)^2}{\sigma^3} + 2\sum_{i=1}^{n} \frac{2\alpha(y_i - \mu)b_i}{\sigma^2 (1+b_i^2)}$$

$$\frac{\partial l(\theta)}{\partial \alpha} = -\frac{n(16\alpha + 12\alpha^3)}{4 + 8\alpha^2 + 3\alpha^4} + 2\sum_{i=1}^{n} -\frac{2(y_i - \mu)b_i}{\sigma(1+b_i^2)}$$

where, $b_i = \left(1 - \frac{\alpha(y_i - \mu)}{\sigma}\right)$.

The solutions of the above system of likelihood equations gives the maximum likelihood estimator for $\theta = (\alpha, \mu, \sigma)$ which can be obtained by numerically maximizing eqn. (26) with respect to the parameters $\theta = (\alpha, \mu, \sigma)$. The derivation of the observed information matrix is also obtained using numerical procedures. Initial values for those procedures can be obtained although the moment estimators.

The log-likelihood function based on a single observation Y for the parameters $\theta = (\alpha, \mu, \sigma)$, is given in Appendix E. The variance-covariance matrix of the MLEs can be obtained by taking the inverse of the Fisher information matrix (**I**) as given in Appendix E.

6. REAL LIFE APPLICATIONS: COMPARATIVE DATA FITTING

Here we have considered two datasets: the dataset 1 is related to N latitude degrees in 69 samples from world lakes, which appear in Column 5 of the Diversity data set in website: http://users.stat.umn.edu/sandy/courses/8061/datasets/lakes.lsp; and the dataset 2 is the body mass index (BMI) of 202 Australian athletes (Cook and Weisberg, 1994).

We then compared the proposed distribution $BASN_2(\alpha, \mu, \sigma)$ with the normal distribution $N(\mu, \sigma^2)$, the logistic distribution $LG(\mu, \beta)$, the Laplace distribution $La(\mu, \beta)$, the skew-normal distribution $SN(\lambda, \mu, \sigma)$ of Azzalini (1985), the skew-logistic distribution $SLG(\lambda, \mu, \beta)$ of Wahed and Ali (2001), the skew-Laplace distribution $SLa(\lambda, \mu, \beta)$ of Nekoukhou and Alamatsaz (2012), the alpha-skew-normal distribution $ASN(\alpha, \mu, \sigma)$ of Olivero (2010), the alpha-skew-Laplace distribution $ASLa(\alpha, \mu, \beta)$ of Harandi and Alamatsaz (2013), the alpha-skew-logistic distribution $ASLG(\alpha, \mu, \beta)$ of Hazarika and Chakraborty (2014), the alpha-beta-skew-normal distribution $ABSN(\alpha, \beta, \mu, \sigma)$ and beta-skew-normal distribution $BSN(\beta, \mu, \sigma)$ of Shafiei et al. (2016). The reason behind the choice of the above distribution for comparison lies in the fact that the Logistic and Laplace distributions and their skewed version are main competitors of the proposed distribution.

Using R software package GenSA package version-1.0.3, (Xiang et al., 2013), the MLE of the parameters are obtained by using numerical optimization routine. AIC and BIC are used for model comparison.

Table 1 and Table 2 shows the MLE's, log-likelihood, AIC and BIC of the above mentioned distributions. The graphical representations of the results taking only the top three competitors for the proposed model are given in Figure 10 and Figure 11.

Table 1. MLE's, log-likelihood, AIC and BIC for N latitude degrees in 69 samples from world lakes.

Parameters→ Distributions ↓	μ	σ	λ	α	β	$\log L$	AIC	BIC
$N(\mu,\sigma^2)$	45.165	9.549	--	--	--	-253.599	511.198	515.666
$LG(\mu,\beta)$	43.639	--	--	--	4.493	-246.645	497.290	501.758
$SN(\lambda,\mu,\sigma)$	35.344	13.70	3.687	--	--	-243.036	492.072	498.774
$BSN(\beta,\mu,\sigma)$	54.47	5.52	--	--	0.74	-242.530	491.060	497.760
$SLG(\lambda,\mu,\beta)$	36.787	--	2.8284	--	6.417	-239.053	490.808	490.808
$La(\mu,\beta)$	43.00	--	--	--	5.895	-239.248	482.496	486.964
$ASLG(\alpha,\mu,\beta)$	49.087	--	--	0.861	3.449	-237.351	480.702	487.404
$SLa(\lambda,\mu,\beta)$	42.30	--	0.255	--	5.943	-236.900	479.799	486.501
$ASLa(\alpha,\mu,\beta)$	42.3	--	--	-0.220	5.440	-236.079	478.159	484.861
$ASN(\alpha,\mu,\sigma)$	52.147	7.714	--	2.042	--	-235.370	476.739	483.441
$ABSN(\alpha,\beta,\mu,\sigma)$	47.69	7.15	--	1.72	-0.37	-230.770	469.530	478.480
$BASN_2(\alpha,\mu,\sigma)$	54.265	6.559	--	1.994	--	**-226.228**	**458.455**	**465.158**

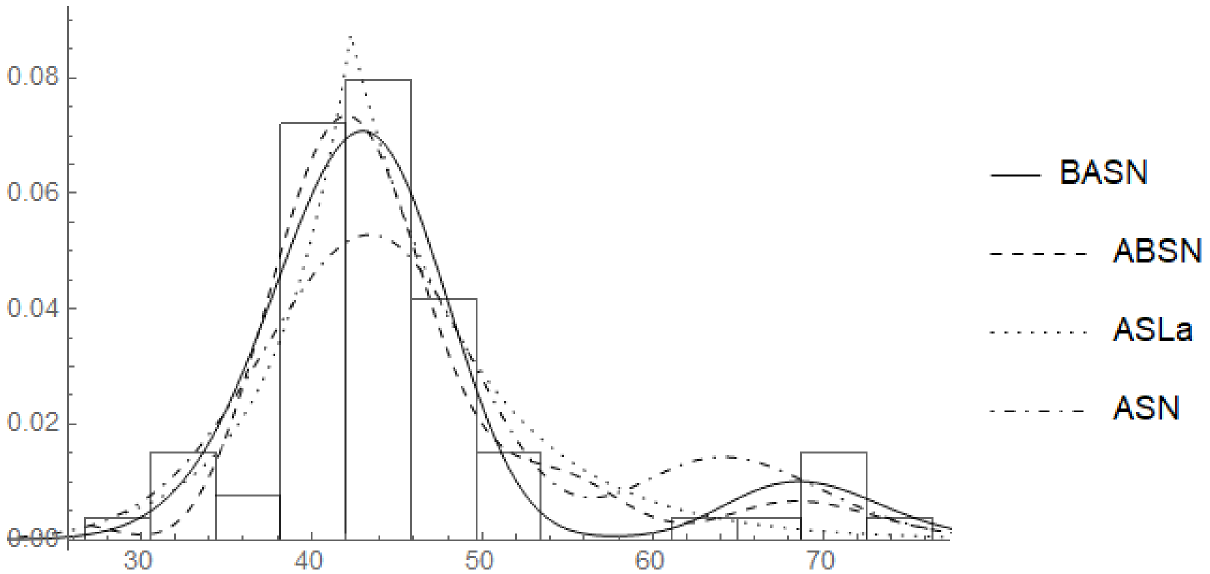


Figure 10. Plots of observed and expected densities of some distributions for N latitude degrees in 69 samples from world lakes.

Table 2. MLE's, log-likelihood, AIC and BIC for body mass index (BMI) of 202 Australian athletes.

Parameters → / Distributions ↓	μ	σ	λ	α	β	$\log L$	AIC	BIC
$N(\mu,\sigma^2)$	22.956	2.857	--	--	--	-498.668	1001.336	1007.953
$La(\mu,\beta)$	22.749	--	--	--	2.123	-494.08	992.16	998.7765
$BSN(\beta,\mu,\sigma)$	22.528	2.694	--	--	-0.058	-492.88	991.76	1001.685
$ASLa(\alpha,\mu,\beta)$	22.350	--	--	-0.14	2.07	-492.601	991.202	1001.127
$SLa(\lambda,\mu,\beta)$	22.350	--	0.865	--	2.084	-492.461	990.922	1000.847
$LG(\mu,\beta)$	22.787	--	--	--	1.529	-491.462	986.924	993.5405
$SN(\lambda,\mu,\sigma)$	19.969	4.133	2.313	--	--	-490.099	986.198	996.1228
$ASLG(\alpha,\mu,\beta)$	21.933	--	--	-0.201	1.475	-489.094	984.188	994.1128
$ASN(\alpha,\mu,\sigma)$	24.834	2.653	--	0.994	--	-488.69	983.38	993.3048
$ABSN(\alpha,\beta,\mu,\sigma)$	23.998	2.853	--	0.817	-0.131	-486.743	981.486	994.7191
$SLG(\lambda,\mu,\beta)$	20.717	--	1.401	--	1.975	-487.311	980.622	990.5468
$BASN_2(\alpha,\mu,\sigma)$	26.482	2.706	--	0.971	--	**-484.773**	**975.546**	**985.4708**

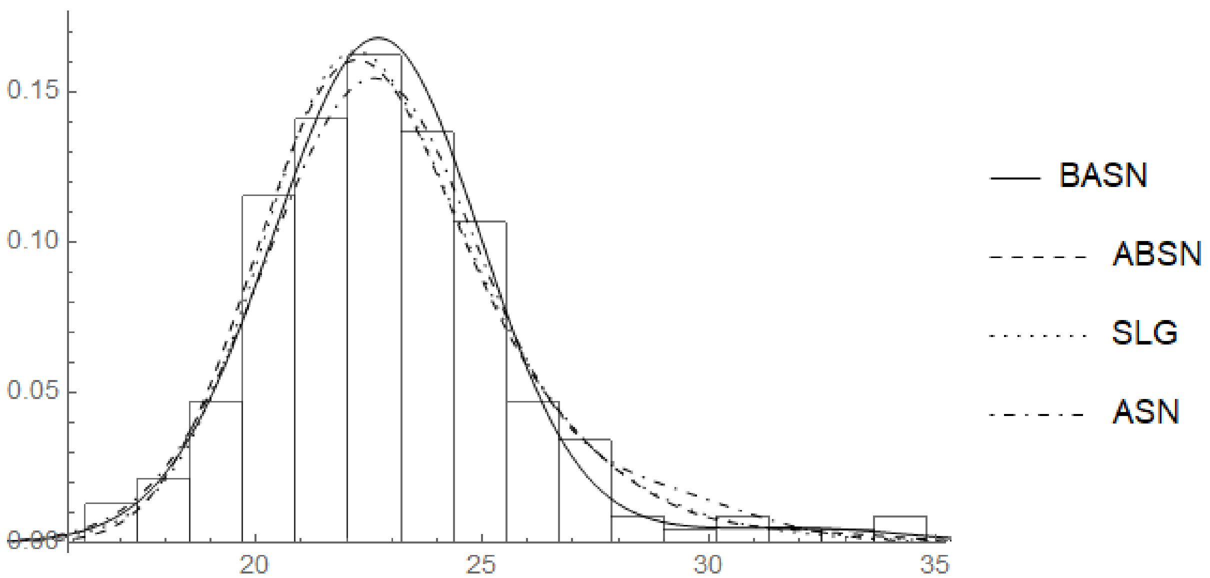


Figure 11. Plots of observed and expected densities of some distributions for body mass index (BMI) of 202 Australian athletes.

It is found from Table 1 and 2 that the proposed $BASN_2(\alpha, \mu, \sigma)$ distribution provides best fit to the data set in terms of AIC and BIC. The plots of observed and expected densities presented in Figure 10 and 11 also confirm our findings.

Remark 10: The observed variance-covariance matrix of the MLEs of the parameters $\theta = (\alpha, \mu, \sigma)$ of $BASN_2(\alpha, \mu, \sigma)$ distribution in dataset 1 and dataset 2 are obtained respectively, as

$$Var - Cov(\hat{\theta}) = \begin{pmatrix} 0.6995 & 0.1615 & 0.1395 \\ 0.1615 & 0.1350 & -0.00455 \\ 0.1395 & -0.00455 & 0.1349 \end{pmatrix} \; \& \; Var - Cov(\hat{\theta}) = \begin{pmatrix} 0.1371 & 0.0327 & 0.0337 \\ 0.0327 & 0.0199 & 0.00192 \\ 0.0337 & 0.00192 & 0.0175 \end{pmatrix}$$

6.1. Likelihood Ratio Test

Furthermore, since $N(\mu, \sigma^2)$ and $BASN_2(\alpha, \mu, \sigma)$ distributions are nested models, the likelihood ratio (LR) test is used to differentiate between them. The LR test is carried out to test the following null hypothesis $H_0 : \alpha = 0$, that is the sample is drawn from $N(\mu, \sigma^2)$; against the alternative $H_1 : \alpha \neq 0$, that is the sample is drawn from $BASN_2(\alpha, \mu, \sigma)$.

The values of LR test statistic for the datasets 1 and 2 are respectively 54.742 and 27.79. Both of which exceed the 99% critical value, that is, 6.635. Thus there is evidence in support of the alternative hypothesis that is, the sampled data comes from $BASN_2(\alpha, \mu, \sigma)$, not from $N(\mu, \sigma^2)$.

7. CONCLUDING REMARKS AND FURTHER SCOPE

In this study a new alpha-skew-normal distribution with one parameter which has both unimodal as well as bimodal shapes is constructed and some of its properties are studied. The bathtub shaped failure rate function is seen in the half $BASN_2(\alpha, \mu, \sigma)$ distribution. Some extensions of the proposed distribution with some of their special cases are presented. Our findings adequately supported the proposed $BASN_2(\alpha, \mu, \sigma)$ distribution as the best fitted one to the datasets under consideration in terms of AIC and BIC. The plots of observed and expected densities presented also confirm our findings.

There is scope of extending the present work by considering the Logistic and the Laplace distributions. Moreover, logarithmic forms and bivariate generalizations can also be considered as future work.

8. ACKNOWLEDGEMENT

The second author would like to acknowledge the fellowship received under the UGC-BSR Research Fellowship in Science scheme 2014-15, Govt. of India.

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APPENDIX

A: Proof of Proposition 1

$$F_Z(z) == \frac{1}{C_2(\alpha)(2+\alpha^2)^2} \int_{-\infty}^{z} \left\{\alpha^4 z^4 - 4\alpha^3 z^3 + 8\alpha^2 z^2 - 8\alpha z + 4\right\}\varphi(z)\, dz$$

$$= \frac{\alpha^4[-\{z(3+z^2)\varphi(z)\} + 3\Phi(z)] - 4\alpha^3[-\{(2+z^2)\ \varphi(z)\}] + 8\alpha^2\{-z\varphi(z) + \Phi(z)\} - 8\alpha(-\varphi(z)) + 4\Phi(z)}{C_2(\alpha)(2+\alpha^2)^2}$$

$$= \frac{\varphi(z)[-3\alpha^4 z - \alpha^4 z^3 + 8\alpha^3 + 4\alpha^3 z^2 - 8\alpha^2 z + 8\alpha]}{C_2(\alpha)(2+\alpha^2)^2} + \frac{\Phi(z)[3\alpha^4 + 8\alpha^2 + 4]}{(3\alpha^4 + 8\alpha^2 + 4)}$$

$$= \Phi(z) + \frac{\alpha[8 - 8\alpha z + 4\alpha^2(2+z^2) - \alpha^3 z(3+z^2)\ ]}{C_2(\alpha)(2+\alpha^2)^2}\varphi(z)\ .$$

B: Proof of Proposition 2

When n is even; $E(Z^n) = \int_{-\infty}^{\infty} \frac{z^n}{C_2(\alpha)} \left\{\frac{(1-\alpha z)^2 + 1}{2+\alpha^2}\right\}^2 \varphi(z) dz$

$$= \frac{1}{C_2(\alpha)(2+\alpha^2)^2} \int_{-\infty}^{\infty} \left(\alpha^4 z^{n+4} - 4\,\alpha^3 z^{n+3} + 8\alpha^2 z^{n+2} - 8\alpha\, z^{n+1} + 4z^n\right)\varphi(z)\, dz$$

$$= \frac{1}{C_2(\alpha)(2+\alpha^2)^2}\left[\alpha^4 \int_{-\infty}^{\infty} z^{n+4}\varphi(z)\, dz + 8\alpha^2 \int_{-\infty}^{\infty} z^{n+2}\varphi(z) dz + 4\int_{-\infty}^{\infty} z^n\ \varphi(z)\, dz\right]$$

using the result of n^{th} order moment of normal distribution in the above equation we get the result in (9). Similarly, when n is odd the same can be obtained with the help of following equation

$$E(Z^n) = \frac{1}{C_2(\alpha)(2+\alpha^2)^2}\left[-4\alpha^3 \int_{-\infty}^{\infty} z^{n+3}\varphi(z) dz - 8\alpha \int_{-\infty}^{\infty} z^{n+1}\varphi(z) dz\right].$$

C: Proof of Proposition 3

$$M_Z(t) == \frac{1}{C_2(\alpha)(2+\alpha^2)^2} \int_{-\infty}^{\infty} e^{tz}\left[\alpha^4 z^4 - 4\alpha^3 z^3 + 8\alpha^2 z^2 - 8\alpha z + 4\right]\varphi(z)\, dz \qquad \text{(A1)}$$

Again, $\int_{-\infty}^{\infty}\varphi(z) e^{tz} dz = e^{\frac{t^2}{2}} = M_X(t)$, i.e., mgf of standard normal variable

$$\int_{-\infty}^{\infty} z\varphi(z) e^{tz} dz = t\, M_X(t),\ \int_{-\infty}^{\infty} z^2\varphi(z) e^{tz} dz = t^2 M_X(t) + M_X(t)$$

$$\int_{-\infty}^{\infty} z^3\varphi(z) e^{tz} dz = t^3 M_X(t) + 3t\, M_X(t) \text{ and } \int_{-\infty}^{\infty} z^4\varphi(z) e^{tz} dz = t^4 M_X(t) + 6t^2 M_X(t) + 3M_X(t)$$

Now, applying the above results in (A1) we get the expression in (10).

D: Proof of Proposition 4

$$F(z)=\frac{1}{(2+\alpha^2)(2+3\alpha^2)}\int_{-\infty}^{z}\alpha^4 z^4+8\alpha^2 z^2+4\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\right]dz$$

$$=\frac{\alpha^4[-z(3+z^2)\varphi(z)+3\Phi(z)]+8\alpha^2[-z\,\varphi(x)+\Phi(z)]+4\Phi(z)}{C_2(\alpha)(2+\alpha^2)^2}$$

On simplifying the above equation we get the result in (12).

Again, $M_Z(t)=\dfrac{1}{C_2(\alpha)(2+\alpha^2)^2}\displaystyle\int_{-\infty}^{\infty}e^{tz}\left[\alpha^4 z^4+8\alpha^2 z^2+4\right]\varphi(z)\,dz$ (A2)

Again, $\displaystyle\int_{-\infty}^{\infty}\varphi(z)e^{tz}dz=e^{\frac{t^2}{2}}=M_X(t),$ i.e., mgf of standard normal variable

$\displaystyle\int_{-\infty}^{\infty}z^2\varphi(z)e^{tz}dz=t^2M_X(t)+M_X(t)$ and $\displaystyle\int_{-\infty}^{\infty}z^4\varphi(z)e^{tz}dz=t^4M_X(t)+6t^2M_X(t)+3M_X(t)$

Now, applying the above results in (A2), we get the expression in (13).

E: Log-Likelihood and Fisher Information Matrix

$$l(\theta;y)=2\log\left[\left\{1-\alpha\left(\frac{y-\mu}{\sigma}\right)\right\}^2+1\right]-\log(2+\alpha^2)-\log\sigma-\log(2+3\alpha^2)-\frac{1}{2}\log(2\pi)-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2$$

Score functions:

The first-order partial derivatives of $l(\theta;y)$ are:

$$\frac{\partial l(\theta;y)}{\partial\mu}=\frac{(y-\mu)}{\sigma^2}+\frac{4\alpha b}{\sigma(1+b^2)}$$

$$\frac{\partial l(\theta;y)}{\partial\sigma}=-\frac{n}{\sigma}+\frac{(y-\mu)^2}{\sigma^3}+\frac{4\alpha(y-\mu)b}{\sigma^2(1+b^2)}$$

$$\frac{\partial l(\theta;y)}{\partial\alpha}=-\frac{n(16\alpha+12\alpha^3)}{4+8\alpha^2+3\alpha^4}-\frac{4(y-\mu)b}{\sigma(1+b^2)}$$

The second-order partial derivatives of $l(\theta;y)$ are:

$$\frac{\partial^2 l(\theta;y)}{\partial\mu^2}=-\frac{1}{\sigma^2}+2\left(\frac{2\alpha^2}{\sigma^2(1+b^2)}-\frac{4\alpha^2b^2}{\sigma^2(1+b^2)^2}\right)$$

$$\frac{\partial^2 l(\theta;y)}{\partial\sigma^2}=\frac{n}{\sigma^2}-\frac{3(y-\mu)^2}{\sigma^4}+2\left(\frac{2\alpha^2(y-\mu)^2}{\sigma^4(1+b^2)}-\frac{4\alpha^2(y-\mu)^2b^2}{\sigma^4(1+b^2)^2}-\frac{4\alpha(y-\mu)b}{\sigma^3(1+b^2)}\right)$$

$$\frac{\partial^2 l(\theta;y)}{\partial\alpha^2}=-n\left(-\frac{(16\alpha+12\alpha^3)^2}{(4+8\alpha^2+3\alpha^4)^2}+\frac{16+36\alpha^2}{4+8\alpha^2+3\alpha^4}\right)+2\left(\frac{2(y-\mu)^2}{\sigma^2(1+b^2)}-\frac{4(y-\mu)^2b^2}{\sigma^2(1+b^2)^2}\right)$$

$$\frac{\partial^2 l(\theta;y)}{\partial\mu\partial\sigma} = -\frac{2(y-\mu)}{\sigma^3} + \frac{4\alpha^2(y-\mu)}{\sigma^3(1+b^2)} - \frac{8\alpha^2(y-\mu)b^2}{\sigma^3(1+b^2)^2} - \frac{4\alpha b}{\sigma^2(1+b^2)}$$

$$\frac{\partial^2 l(\theta;y)}{\partial\mu\partial\alpha} = -\frac{4\alpha(y-\mu)}{\sigma^2(1+b^2)} + \frac{8\alpha(y-\mu)b^2}{\sigma^2(1+b^2)^2} + \frac{4b}{\sigma(1+b^2)}$$

$$\frac{\partial^2 l(\theta;y)}{\partial\sigma\partial\alpha} = -\frac{4\alpha(y-\mu)^2}{\sigma^3(1+b^2)} + \frac{8\alpha(y-\mu)^2 b^2}{\sigma^3(1+b^2)^2} + \frac{4(y-\mu)b}{\sigma^2(1+b^2)}$$

where, $b = \left(1 - \frac{\alpha(y-\mu)}{\sigma}\right)$.

$$\mathrm{I} = \begin{bmatrix} E\left(-\frac{\partial^2 l(\theta;y)}{\partial\mu^2}\right) & E\left(-\frac{\partial^2 l(\theta;y)}{\partial\mu\partial\sigma}\right) & E\left(-\frac{\partial^2 l(\theta;y)}{\partial\mu\partial\alpha}\right) \\ E\left(-\frac{\partial^2 l(\theta;y)}{\partial\sigma\partial\mu}\right) & E\left(-\frac{\partial^2 l(\theta;y)}{\partial\sigma^2}\right) & E\left(-\frac{\partial^2 l(\theta;y)}{\partial\sigma\partial\alpha}\right) \\ E\left(-\frac{\partial^2 l(\theta;y)}{\partial\alpha\partial\mu}\right) & E\left(-\frac{\partial^2 l(\theta;y)}{\partial\alpha\partial\sigma}\right) & E\left(-\frac{\partial^2 l(\theta;y)}{\partial\alpha^2}\right) \end{bmatrix}^{-1}$$

where, $E\left(-\frac{\partial^2 l(\theta;y)}{\partial\mu^2}\right) \approx -\left.\frac{\partial^2 l(\theta;y)}{\partial\mu^2}\right|_{\mu=\hat{\mu}}$, $E\left(-\frac{\partial^2 l(\theta;y)}{\partial\mu\partial\alpha}\right) \approx -\left.\frac{\partial^2 l(\theta;y)}{\partial\mu\partial\alpha}\right|_{\mu=\hat{\mu},\alpha=\hat{\alpha}}$ etc.