MALAYSIAN JOURNAL OF SCIENCE
MJS
ISSN 1394-3065

VIRTUAL SYMPOSIUM ON MULTIDISCIPLINARY SCIENCE 2021 Special Issue

Malaysian Journal of Science 41 (Special Issue 1): 34-39 (September 2022) https://mjs.um.edu.my

DEGREE SUM ENERGY OF NON-COMMUTING GRAPH FOR DIHEDRAL GROUPS

Mamika Ujianita Romdhini[1ac], Athirah Nawawi[2ab]*

Abstract: For a finite group G, let $Z(G)$ be the centre of G. Then the non-commuting graph on G, denoted by Γ_G, has $G\backslash Z(G)$ as its vertex set with two distinct vertices v_p and v_q joined by an edge whenever $v_p v_q \neq v_q v_p$. The degree sum matrix of a graph is a square matrix whose (p,q)-th entry is $d_{v_p} + d_{v_q}$ whenever p is different from q, otherwise, it is zero, where d_{v_i} is the degree of the vertex v_i. This study presents the general formula for the degree sum energy, $E_{DS}(\Gamma_G)$, for the non-commuting graph of dihedral groups of order $2n$, D_{2n}, for all $n \geq 3$.

Keywords: *Non-commuting graph, dihedral group, degree sum matrix, the energy of a graph.*

1. Introduction

The non-commuting graph on G, denoted by Γ_G, has $G\backslash Z(G)$ as its vertex set with two distinct vertices v_p and v_q joined by an edge whenever $v_p v_q \neq v_q v_p$ (Abdollahi, 2006). In that sense, the non-commuting graph on G, Γ_G can further be associated with the adjacency matrix. The $n \times n$ adjacency matrix $A(\Gamma_G) = [a_{ij}]$ of Γ_G has entries $a_{ij} = 1$ if there is an edge between v_i to v_j, and $a_{ij} = 0$ otherwise. Since Γ_G is a simple graph, then $A(\Gamma_G)$ is a symmetric matrix with zero diagonal entries. For a real number λ, the characteristic polynomial $P_{A(\Gamma_G)}(\lambda)$ of Γ_G is defined by $\det(\lambda I_n - A(\Gamma_G))$, where I_n is an $n \times n$ identity matrix. The eigenvalues of Γ_G are the roots of the equation $P_{A(\Gamma_G)}(\lambda) = 0$, and they are labelled as $\lambda_1, \lambda_2, \ldots, \lambda_n$. The spectrum of Γ_G is given as a list of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, with their respective multiplicities $k_1, k_2, \ldots, k_m$ as exponents, denoted by $Spec(\Gamma_G) = \left\{\lambda_1^{(k_1)}, \lambda_2^{(k_2)}, \ldots, \lambda_m^{(k_m)}\right\}$. Furthermore, for all finite graphs, Gutman (1978) defined the energy of Γ_G as the sum of the absolute values of the eigenvalues, denoted by $E(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|$.

There are several interesting studies regarding the non-commuting graph involving the spectrum and energy of its adjacency matrix. Mahmoud *et al.* (2017) described the adjacency energy of the non-commuting graph for dihedral groups of order $2n$. In the same year, Dutta and Nath (2017) computed the Laplacian energy of the non-commuting graph for finite non-abelian groups, including the dihedral groups of order $2n$. Alternatively, Fasfous and Nath (2020) computed the spectrum and energy of the non-commuting graph for certain classes of finite groups inclusive of D_{2n}. They found that the adjacency energy of the non-commuting graph is not equal to the Laplacian energy for some finite groups. This refutes the conjecture by Gutman *et al.* in 2008, stating that the adjacency energy of any graph is smaller than or equal to its Laplacian energy, which holds for all graphs. However, readers can also see different perspectives of this particular graph where the discussion on the detour index, eccentric connectivity, total eccentricity polynomials, and mean distance of the non-commuting graph for the dihedral group by Khasraw *et al.* (2020).

Throughout this paper, the discussion will be directed to the degree sum energy defined by Ramane *et al.* (2013). In particular, Jog and Kotambari (2016) presented the degree sum energy of six types of simple graphs, namely, Wheel graphs, Path Tadpole graphs, Dumbbell graphs, coalescence regular graphs, complete graphs, and cycles. Apart from that, Hosamani and Ramane (2016) also discussed the degree sum energy focusing on determining the lower bounds of degree sum energy of simple graphs. However, a limited number of studies central to the degree sum matrices for non-commuting graphs have been found. Therefore, we aim to formulate the degree sum energy of the non-commuting graph for the dihedral groups.

For $n \geq 3$, the non-abelian dihedral group D_{2n} of order $2n$ is defined as the reflection and rotation motions that return a regular n-gon to its original state, with the composition operation denoted by D_{2n}. The n rotations are

Authors information:
[a]Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, MALAYSIA.
[b]Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, MALAYSIA.
[c]Department of Mathematics, Faculty of Mathematics and Natural Science, Universitas Mataram, Mataram, 83125, INDONESIA. E-mail: mamika@unram.ac.id[1]

*Corresponding Author: athirah@upm.edu.my

Received: January 21, 2022
Accepted: April 9, 2022
Published: September 30, 2022

DOI: https://doi.org/10.22452/mjs.sp2022no.1.5
Malaysian Journal of Science 41 (Special Issue 1): 34-39 (September 2022)

$\boldsymbol{a^i}$ and the reflections are $\boldsymbol{a^i b}$, where $\boldsymbol{1 \le i \le n}$. Therefore, $\boldsymbol{D_{2n}}$ can be written as:

$$\boldsymbol{D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle}.$$

The centre of $\boldsymbol{D_{2n}}$, $\boldsymbol{Z(D_{2n})}$ is equal to $\boldsymbol{\{e\}}$ **if n is odd** and $\boldsymbol{\{e, a^{\frac{n}{2}}\}}$ **if n is even**. The centralizer of the element $\boldsymbol{a^i}$ in the group $\boldsymbol{D_{2n}}$ is $\boldsymbol{C_{D_{2n}}(a^i) = \{a^i : 1 \le i \le n\}}$ and for the element $\boldsymbol{a^i b}$ is either $\boldsymbol{C_{D_{2n}}(a^i b) = \{e, a^i b\}}$, **if n is odd**, or $\boldsymbol{C_{D_{2n}}(a^i b) = \{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\}}$, **if n is even.**

2. Preliminaries

We define $\boldsymbol{d_{v_p}}$ as the degree of a vertex $\boldsymbol{v_p}$, which is the number of vertices adjacent to $\boldsymbol{v_p}$. The definition of the degree sum matrix is given as follows:

Definition 2.1. (Ramane *et al.,* 2013) The degree sum matrix of order $\boldsymbol{n \times n}$ associated with a graph $\boldsymbol{\Gamma}$ is given by $\boldsymbol{DS(\Gamma) = [ds_{pq}]}$ whose $\boldsymbol{(p,q)}$-th entry is given by

$$ds_{pq} = \begin{cases} d_{v_p} + d_{v_q}, & \text{if } p \neq q \\ 0, & \text{if } p = q \end{cases}$$

In this section, we include some previous results, which benefit the computations of our main results. Recall that, for any $\boldsymbol{n \ge 3}$, $\boldsymbol{D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle}$. We define $\boldsymbol{G_1 = \{a^i : 1 \le i \le n\} \backslash Z(D_{2n})}$ and $\boldsymbol{G_2 = \{a^i b : 1 \le i \le n\}}$. The following is the result of the degree of each vertex in the non-commuting graph of $\boldsymbol{G = G_1 \cup G_2}$.

Theorem 2.1: (Khasraw *et al.*, 2020) Let $\boldsymbol{\Gamma_G}$ be the non-commuting graph on $\boldsymbol{G}$, where $\boldsymbol{G = G_1 \cup G_2}$. Then,

1. $d_{a^i} = n$, and
2. $d_{a^i b} = \begin{cases} 2n - 2, & \text{if } n \text{ is odd} \\ 2n - 4, & \text{if } n \text{ is even} \end{cases}$.

A graph which has $\boldsymbol{n}$ vertices with the degree of every vertex being $\boldsymbol{n-1}$ is called a complete graph $\boldsymbol{K_n}$. Moreover, the complement of the complete graph $\boldsymbol{K_n}$ is written as $\boldsymbol{\overline{K}_n}$. Consequently, the isomorphism of the non-commuting graph with some common types of graphs can be seen in the following result:

Theorem 2.2: (Khasraw *et al.,* 2020) Let $\boldsymbol{\Gamma_G}$ be the non-commuting graph on $\boldsymbol{D_{2n}}$.

1. If $\boldsymbol{G = G_1}$, then $\boldsymbol{\Gamma_G \cong \overline{K}_m}$, where $\boldsymbol{m = |G_1|}$.
2. If $\boldsymbol{G = G_2}$, then $\boldsymbol{\Gamma_G \cong \begin{cases} K_n, & \textbf{if } n \textbf{ is odd} \\ K_n - \frac{n}{2}K_2, & \textbf{if } n \textbf{ is even} \end{cases}}$,

where $\boldsymbol{\frac{n}{2}K_2}$ denotes $\boldsymbol{\frac{n}{2}}$ copies of $\boldsymbol{K_2}$.

The following lemma helps us to compute the characteristic polynomial of the non-commuting graph of $\boldsymbol{D_{2n}}$.

Lemma 2.1: (Ramane & Shinde, 2017) If $\boldsymbol{a, b, c}$ and $\boldsymbol{d}$ are real numbers and $\boldsymbol{J_n}$ is an $\boldsymbol{n \times n}$ matrix whose entries are equal to one, then the determinant of the $\boldsymbol{(n_1 + n_2) \times (n_1 + n_2)}$ matrix of the form

$$\boldsymbol{\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}},$$

can be simplified in an expression given by

$\boldsymbol{(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}\big((\lambda - (n_1 - 1)a)(\lambda - (n_2 - 1)b) - n_1 n_2 cd\big)}$,

where $\boldsymbol{1 \le n_1, n_2 \le n}$ and $\boldsymbol{n_1 + n_2 = n}$.

The following lemma is the result of the spectrum of the complete graph, which is useful for computing the energy of the non-commuting graph for $\boldsymbol{D_{2n}}$.

Lemma 2.2: (Brouwer & Haemers, 2010) If $\boldsymbol{K_n}$ is the complete graph on $\boldsymbol{n}$ vertices, then its adjacency matrix is $\boldsymbol{J_n - I_n}$ and the spectrum of $\boldsymbol{K_n}$ is $\boldsymbol{\{(n-1)^{(1)}, (-1)^{(n-1)}\}}$.

3. Main Results

This section presents several results on the degree sum energy of the non-commuting graph on the dihedral group of order $\boldsymbol{2n}$, $\boldsymbol{D_{2n}}$.

Theorem 3.1. Let $\boldsymbol{\Gamma_G}$ be the non-commuting graph on $\boldsymbol{G}$ and $\boldsymbol{E_{DS}}$ be the degree sum energy of $\boldsymbol{\Gamma_G}$.
1. If $\boldsymbol{G = G_1}$, then $\boldsymbol{E_{DS}(\Gamma_G) = 0}$.
2. If $\boldsymbol{G = G_2}$, then

$$\boldsymbol{E_{DS}(\Gamma_G) = \begin{cases} 4(n-1)^2, & \textbf{if } n \textbf{ is odd} \\ 4(n-2)(n-1), & \textbf{if } n \textbf{ is even} \end{cases}}.$$

Proof.
1. **When n is odd.** From Theorem 2.2 (1), $\boldsymbol{\Gamma_G = \overline{K}_m}$, where $\boldsymbol{G = G_1}$ and $\boldsymbol{m = |G_1| = n - 1}$. Then, every vertex of $\boldsymbol{\Gamma_G}$ has degree zero. Thus, the degree sum matrix of $\boldsymbol{\Gamma_G}$ is an $\boldsymbol{(n-1) \times (n-1)}$ zero matrix, $\boldsymbol{DS(\Gamma_G) = [0]}$. The only eigenvalue of $\boldsymbol{DS(\Gamma_G)}$ is zero with multiplicity $\boldsymbol{n-1}$. Thus, $\boldsymbol{E_{DS}(\Gamma_G) = 0}$.

   When n is even. From Theorem 2.2 (1), $\boldsymbol{\Gamma_G = \overline{K}_m}$, where $\boldsymbol{G = G_1}$ and $\boldsymbol{m = |G_1| = n - 2}$, removing $\boldsymbol{e}$ and $\boldsymbol{a^{\frac{n}{2}}}$ in $\boldsymbol{Z(D_{2n})}$. Then, every vertex of $\boldsymbol{\Gamma_G}$ has degree zero. Hence, the degree sum matrix of $\boldsymbol{\Gamma_G}$ is an $\boldsymbol{(n-2) \times (n-2)}$ zero matrix, $\boldsymbol{DS(\Gamma_G) = [0]}$. The only eigenvalue of $\boldsymbol{DS(\Gamma_G)}$ is zero with multiplicity $\boldsymbol{n-2}$. Thus, $\boldsymbol{E_{DS}(\Gamma_G) = 0}$.

DOI: https://doi.org/10.22452/mjs.sp2022no.1.5
Malaysian Journal of Science 41 (Special Issue 1): 34-39 (September 2022)

2. When n is odd. From Theorem 2.2 (2), $\Gamma_G = K_n$, where $G = G_2$. Then, every vertex has a degree $n-1$. Thus, the

$$DS(\Gamma_G) = \begin{bmatrix} 0 & 2(n-1) & \cdots & 2(n-1) \\ 2(n-1) & 0 & \cdots & 2(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ 2(n-1) & 2(n-1) & \cdots & 0 \end{bmatrix}$$
$$= 2(n-1)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

degree sum matrix of Γ_G is an $n \times n$ matrix, $DS(\Gamma_G) = [ds_{pq}]$ whose (p,q)-th entry is $ds_{pq} = (n-1)+(n-1) = 2(n-1)$ for $p \neq q$, and 0 otherwise. Hence,
In other words, the degree sum matrix of Γ_G is the product of $2(n-1)$ and the adjacency matrix of K_n. Based on Lemma 2.2, $Spec(K_n)$ is given by $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$. Since the adjacency energy of K_n is $|n-1| + (n-1)|-1| = 2(n-1)$, the degree sum energy of Γ_G will be $2(n-1) \cdot 2(n-1) = 4(n-1)^2$.

When n is even. From Theorem 2.2 (2), $\Gamma_G = K_n - \frac{n}{2}K_2$, where $G = G_2$. Then, every vertex has a degree of $n-2$. We can now construct an $n \times n$ degree sum matrix of Γ_G,

$$DS(\Gamma_G) = \begin{bmatrix} 0 & 2(n-2) & \cdots & 2(n-2) \\ 2(n-2) & 0 & \cdots & 2(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ 2(n-2) & 2(n-2) & \cdots & 0 \end{bmatrix}$$
$$= 2(n-2)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}.$$

$DS(\Gamma_G) = [ds_{pq}]$ whose (p,q)-th entry is $ds_{pq} = n-2+n-2 = 2(n-2)$ for $p \neq q$, and 0 otherwise. Hence,

In other words, the degree sum matrix of Γ_G is the product of $2(n-2)$ and the adjacency matrix of K_n. Using the same argument as in the previous case, the degree sum energy of Γ_G is given by $2(n-2) \cdot 2(n-1) = 4(n-2)(n-1)$.

The illustration of Theorem 3.1 is given by the following examples for $n = 4$ and $n = 5$.

Example 1. Let Γ_G be the non-commuting graph on G, where $G \subset D_8$, $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, $Z(D_8) = \{e, a^2\}$, $G_1 = \{a, a^3\}$, $G_2 = \{b, ab, a^2b, a^3b\}$, $C_{D_{2n}}(b) = \{e, a^2, b, a^2b\} = C_{D_{2n}}(a^2b)$, $C_{D_{2n}}(ab) = \{e, a^2, ab, a^3b\} = C_{2n}(a^3b)$. By using the information on the centralizer of each element in G, then the non-commuting graph of G is given as in Figure 1.

When $G = G_1$ from Figure 1 (i), it is clear that we only have two vertices a and a^3 and the degree of each vertex is zero. Then, the non-commuting graph of G_1 is the complement of the complete graph on two vertices, $\overline{K}_2$. This implies that we have a 2×2 degree sum matrix of Γ_G with all the entries are zero, $DS(\Gamma_G) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Furthermore, the characteristic polynomial of $DS(\Gamma_G)$ is $P_{DS(\Gamma_G)}(\lambda) = \det(\lambda I_2 - DS(\Gamma_G)) = \lambda^2$. It follows that the eigenvalues of $DS(\Gamma_G)$ is zero with multiplicity 2. Therefore, the degree sum energy of Γ_G is $E_{DS}(\Gamma_G) = 0$.

However, if $G = G_2$, then each vertex $a^i b$, where $1 \leq i \leq 4$, is of degree two, as shown in Figure 1 (ii). Then, the non-commuting graph of G_2 on four vertices is $K_4 - 2K_2$. This means that we have a 4×4 degree sum matrix of Γ_G with the non-diagonal entries are $2+2=4$, while the diagonal entries are zero. Then, we obtain

$$DS(\Gamma_G) = \begin{bmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 \\ 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

.

Furthermore, the characteristic polynomial of $DS(\Gamma_G)$ is $P_{DS(\Gamma_G)}(\lambda) = \det(\lambda I_4 - DS(\Gamma_G)) = (\lambda+4)^3(\lambda-12)$. This implies that the eigenvalues of $DS(\Gamma_G)$ are a single $\lambda = 12$ and $\lambda = -4$ with multiplicity 3. Therefore, $E_{DS}(\Gamma_G) = |12| + 3|-4| = 24 = 4(4-2)(4-1)$.

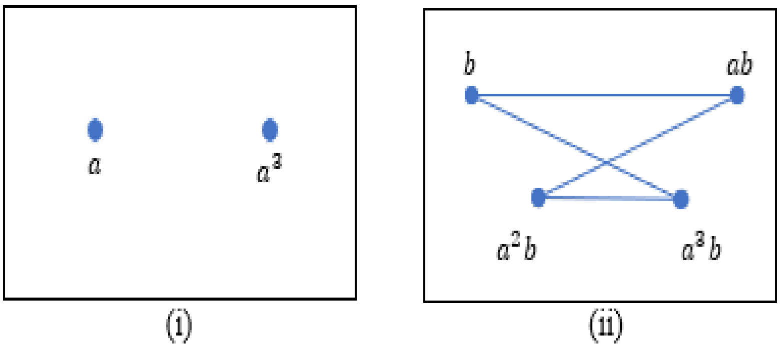

Figure 1. Non-commuting graph of G, where (i) $G = G_1$ and (ii) $G = G_2$.

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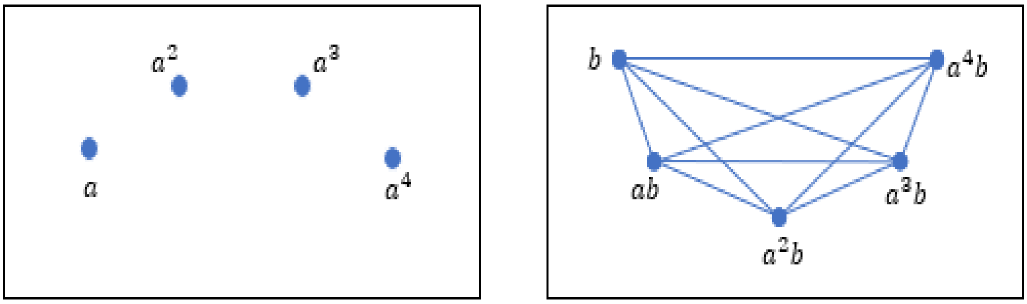

(i) (ii)

Figure 2. Non-commuting graph of $\boldsymbol{G}$, where (i) $\boldsymbol{G = G_1}$, and (ii) $\boldsymbol{G = G_2}$.

Example 2. Let $\boldsymbol{\Gamma_G}$ be the commuting graph on $\boldsymbol{G}$, where $\boldsymbol{G \subset D_{10}}$, $\boldsymbol{D_{10} = \{e,\ a,\ a^2,\ a^3,\ a^4\ b,\ ab,\ a^2b,\ a^3b,\ a^4b\}}$, $\boldsymbol{Z(D_{10}) = \{e\}}$, $\boldsymbol{G_1 = \{a, a^2, a^3, a^4\}}$, $\boldsymbol{G_2 = \{\ b,\ ab,\ a^2b,\ a^3b,\ a^4b\}}$, $\boldsymbol{C_{D_{2n}}(a^ib) = \{e, a^ib\}}$, and $\boldsymbol{C_{D_{2n}}(a^i) = \{a^i \colon 1 \le i \le n\}}$. Using the information on the centralizer of each element in $\boldsymbol{G}$, the non-commuting graph of $\boldsymbol{G}$ is given in Figure 2. When $\boldsymbol{G = G_1}$, from Figure 2 (i), it is clear that we have four vertices $\boldsymbol{a^i}$, for $\boldsymbol{1 \le i \le 4}$, and the degree of each vertex is zero. Then the non-commuting graph of $\boldsymbol{G_1}$ is the complement of the complete graph on four vertices, $\boldsymbol{\overline{K}_4}$. This implies that we have a $\boldsymbol{4 \times 4}$ degree sum matrix of $\boldsymbol{\Gamma_G}$ with all the entries are zero, $\boldsymbol{DS(\Gamma_G) = [0]}$. Furthermore, the characteristic polynomial of $\boldsymbol{DS(\Gamma_G)}$ is $\boldsymbol{P_{DES(\Gamma_G)}(\lambda) = \det(\lambda I_4 - DS(\Gamma_G)) = \lambda^4}$. It follows that the eigenvalues of $\boldsymbol{DS(\Gamma_G)}$ is zero with multiplicity $\boldsymbol{4}$. Therefore, the degree sum energy of $\boldsymbol{\Gamma_G}$ is $\boldsymbol{E_{DS}(\Gamma_G) = 0}$.

In another case, if $\boldsymbol{G = G_2}$, with each vertex $\boldsymbol{a^ib}$, where $\boldsymbol{1 \le i \le 5}$, is of degree four as shown in Figure 2 (ii), then the non-commuting graph of $\boldsymbol{G_2}$ on five vertices is the complete graph, $\boldsymbol{K_5}$. This implies that we have a $\boldsymbol{5 \times 5}$ degree sum matrix of $\boldsymbol{\Gamma_G}$ with the non-diagonal entries are $\boldsymbol{4 + 4 = 8}$, while the diagonal entries are zero. Then, we obtain

$$\boldsymbol{DS(\Gamma_G) = \begin{bmatrix} 0 & 8 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 & 8 \\ 8 & 8 & 0 & 8 & 8 \\ 8 & 8 & 8 & 0 & 8 \\ 8 & 8 & 8 & 8 & 0 \end{bmatrix}}.$$

Furthermore, the characteristic polynomial of $\boldsymbol{\Gamma_G}$ is $\boldsymbol{P_{DS(\Gamma_G)}(\lambda) = \det(\lambda I_5 - DS(\Gamma_G)) = (\lambda + 8)^4(\lambda - 32)}$. This implies that the eigenvalues of $\boldsymbol{DS(\Gamma_G)}$ are a single $\boldsymbol{\lambda = 32}$ and $\boldsymbol{\lambda = -8}$ with multiplicity 4. Therefore, $\boldsymbol{E_{DS}(\Gamma_G) = |32| + 4|-8| = 64 = 4(5-1)^2}$.

Theorem 3.2. Let $\boldsymbol{\Gamma_G}$ be the non-commuting graph on $\boldsymbol{G}$, where $\boldsymbol{G = G_1 \cup G_2 \subset D_{2n}}$, then the characteristic polynomial of degree sum matrices for $\boldsymbol{\Gamma_G}$ is given by

1. $\boldsymbol{P_{DS(\Gamma_G)}(\lambda) = (\lambda + 2n)^{n-2}(\lambda + 2(2n-2))^{n-1}(\lambda^2 - 2(3n^2 - 6n + 2)\lambda - n(n-1)(n^2 + 12n - 12)}$, for $\boldsymbol{n}$ is odd, and
2. $\boldsymbol{P_{DS(\Gamma_G)}(\lambda) = (\lambda + 2n)^{n-3}(\lambda + 2(2n-4))^{n-1}(\lambda^2 - 2(3n^2 - 9n + 4)\lambda - n(n^3 + 6n^2 - 24n + 16)}$, for $\boldsymbol{n}$ is even.

Proof.

1. By Theorem 2.1 for the odd $\boldsymbol{n}$ case, we have $\boldsymbol{d_{a^i} = n}$ and $\boldsymbol{d_{a^ib} = 2n - 2}$, for all $\boldsymbol{1 \le i \le n}$. Then, using the fact that $\boldsymbol{Z(D_{2n}) = \{e\}}$, we have $\boldsymbol{2n - 1}$ vertices for $\boldsymbol{\Gamma_G}$, where $\boldsymbol{G = G_1 \cup G_2}$. The set of vertices consists of $\boldsymbol{n - 1}$ vertices of $\boldsymbol{a^i}$, for $\boldsymbol{1 \le i \le n - 1}$, and $\boldsymbol{n}$ vertices of $\boldsymbol{a^ib}$, for $\boldsymbol{1 \le i \le n}$. Then, the degree sum matrix for $\boldsymbol{\Gamma_G}$ is a $\boldsymbol{(2n-1) \times (2n-1)}$ matrix, $\boldsymbol{DS(\Gamma_G) = [ds_{pq}]}$ whose $\boldsymbol{(p, q)}$-th entries are:
   (i) $\boldsymbol{ds_{pq} = n + n = 2n}$, for $\boldsymbol{p \ne q}$, and $\boldsymbol{1 \le p, q \le n - 1}$,
   (ii) $\boldsymbol{ds_{pq} = n + (2n-2) = 3n - 2}$, for $\boldsymbol{1 \le p \le n - 1}$ and $\boldsymbol{n \le q \le 2n - 1}$,
   (iii) $\boldsymbol{ds_{pq} = (2n-2) + n = 3n - 2}$, for $\boldsymbol{n \le p \le 2n - 1}$ and $\boldsymbol{1 \le q \le n - 1}$,
   (iv) $\boldsymbol{ds_{pq} = (2n-2) + (2n-2) = 2(2n-2)}$, for $\boldsymbol{p \ne q}$, $\boldsymbol{n \le p, q \le 2n - 1}$,
   (v) $\boldsymbol{ds_{pq} = 0}$, for $\boldsymbol{p = q}$.

   We can construct $\boldsymbol{DS(\Gamma_G)}$ given as follows:

$$\boldsymbol{DS(\Gamma_G) = \left[\begin{array}{cccc|cccc} 0 & 2n & \cdots & 2n & 3n-2 & 3n-2 & \cdots & 3n-2 \\ 2n & 0 & \cdots & 2n & 3n-2 & 3n-2 & \cdots & 3n-2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n & 2n & \cdots & 0 & 3n-2 & 3n-2 & \cdots & 3n-2 \\ \hline 3n-2 & 3n-2 & \cdots & 3n-2 & 0 & 2(2n-2) & \cdots & 2(2n-2) \\ 3n-2 & 3n-2 & \cdots & 3n-2 & 2(2n-2) & 0 & \cdots & 2(2n-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n-2 & 3n-2 & \cdots & 3n-2 & 2(2n-2) & 2(2n-2) & \cdots & 0 \end{array}\right]}$$

$$\boldsymbol{= \begin{bmatrix} 2n(J_{n-1} - I_{n-1}) & (3n-2)J_{(n-1)\times n} \\ (3n-2)J_{n\times(n-1)} & 2(2n-2)(J_n - I_n) \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}}.$$

   In this case, $\boldsymbol{DS(\Gamma_G)}$ is divided into four blocks, where the first block is $\boldsymbol{B_1}$, which is a block of $\boldsymbol{(n-1) \times (n-1)}$ matrix with zero diagonal, and every non-diagonal entry is $\boldsymbol{2n}$. In the next two blocks, we have $\boldsymbol{B_2}$ and $\boldsymbol{B_3}$ matrices, which are of the size $\boldsymbol{(n-1) \times n}$ and $\boldsymbol{n \times (n-1)}$, respectively, whose entries are $\boldsymbol{3n - 2}$. The last block

DOI: https://doi.org/10.22452/mjs.sp2022no.1.5
Malaysian Journal of Science 41 (Special Issue 1): 34-39 (September 2022)

is $\boldsymbol{B_4}$, which is an $\boldsymbol{n \times n}$ matrix with zero diagonal, and every non-diagonal entry is $\boldsymbol{2(2n-2)}$. Then, we obtain the characteristic polynomial of $\boldsymbol{DS(\Gamma_G)}$ from the following determinant

$$P_{DS(\Gamma_G)}(\lambda) = |\lambda I_{2n-1} - DS(\Gamma_G)|$$
$$= \begin{vmatrix} (\lambda+2n)I_{n-1} - 2nJ_{n-1} & -(3n-2)J_{(n-1)\times n} \\ -(3n-2)\,J_{n\times(n-1)} & (\lambda+2(2n-2))I_n - 2(2n-2)J_n \end{vmatrix}.$$

Using Lemma 2.1, with $a = 2n$, $b = 2(2n-2)$, $c = 3n-2$, $d = 3n-2$, $n_1 = n-1$ and $n_2 = n$, we obtain the required result.

2. Again, by Theorem 2.1 for the even n case, we know that $d_{a^i} = n$ and $d_{a^i b} = 2n-4$, for all $1 \le i \le n$. Then, using the fact that $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, we have $2n-2$ vertices for Γ_G, where $G = G_1 \cup G_2$. The set of vertices consists of $n-2$ vertices of a^i, for $1 \le i \le n-1$, $i \neq \frac{n}{2}$, and n vertices of $a^i b$, for $1 \le i \le n$. Then, the degree sum matrix for Γ_G is a $(2n-2)\times(2n-2)$ matrix, $DS(\Gamma_G) = [ds_{pq}]$ whose (p,q)-th entry is
(i) $ds_{pq} = n + n = 2n$, for $p \neq q$, and $1 \le p, q \le n-2$,
(ii) $ds_{pq} = n + (2n-4) = 3n-4$, for $1 \le p \le n-2$ and $n-1 \le q \le 2n-2$,
(iii) $ds_{pq} = (2n-4) + n = 3n-4$, for $n-1 \le p \le 2n-2$ and $1 \le q \le n-2$,
(iv) $ds_{pq} = (2n-4) + (2n-4) = 2(2n-4)$, for $p \neq q$, $n-1 \le p, q \le 2n-2$,
(v) $des_{pq} = 0$, for $p = q$.

We can construct $DS(\Gamma_G)$ as follows:

$$DS(\Gamma_G) = \left[\begin{array}{cccc|cccc} 0 & 2n & \cdots & 2n & 3n-4 & 3n-4 & \cdots & 3n-4 \\ 2n & 0 & \cdots & 2n & 3n-4 & 3n-4 & \cdots & 3n-4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n & 2n & \cdots & 0 & 3n-4 & 3n-4 & \cdots & 3n-4 \\ \hline 3n-4 & 3n-4 & \cdots & 3n-4 & 0 & 2(2n-4) & \cdots & 2(2n-4) \\ 3n-4 & 3n-4 & \cdots & 3n-4 & 2(2n-4) & 0 & \cdots & 2(2n-4) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n-4 & 3n-4 & \cdots & 3n-4 & 2(2n-4) & 2(2n-4) & \cdots & 0 \end{array}\right]$$
$$= \begin{bmatrix} 2n(J_{n-2} - I_{n-2}) & (3n-4)J_{(n-2)\times n} \\ (3n-4)J_{n\times(n-2)} & 2(2n-4)(J_n - I_n) \end{bmatrix}$$
$$= \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}.$$

In this case, $DS(\Gamma_G)$ is divided into four blocks, where the first block is M_1, which is a block of $(n-2)\times(n-2)$ matrix with zero diagonal, where every non-diagonal entry is $2n$. The next two blocks are M_2 and M_3, which are of the size $(n-2)\times n$ and $n \times (n-2)$, respectively, whose all entries are equal to $3n-4$. The last block is M_4, which is an $n \times n$ matrix with zero diagonal, while every non-diagonal entry is $2(2n-4)$. Then, we obtain the characteristic polynomial of $DS(\Gamma_G)$ from the following determinant

$$P_{DS(\Gamma_G)}(\lambda) = |\lambda I_{2n-2} - DS(\Gamma_G)|$$
$$= \begin{vmatrix} (\lambda+2n)I_{n-2} - 2nJ_{n-2} & -(3n-4)J_{(n-2)\times n} \\ -(3n-4)\,J_{n\times(n-2)} & (\lambda+2(2n-4))I_n - 2(2n-4)J_n \end{vmatrix}.$$

Using Lemma 2.1, with $a = 2n$, $b = 2(2n-4)$, $c = 3n-4$, $d = 3n-4$, $n_1 = n-2$ and $n_2 = n$, we obtain the required result.

Consequently, the degree sum energy of the non-commuting graph for the dihedral group of order $2n$ can be expressed in the following theorem.

Theorem 3.3: Let Γ_G be the non-commuting graph on G, where $G = G_1 \cup G_2$, then the degree sum energy for Γ_G is given by

1. for n is odd,
$$E_{DS}(\Gamma_G) = 2(3n^2 - 6n + 2) + 2\sqrt{10n^4 - 25n^3 + 24n^2 - 12n + 4},$$

2. and for n is even,
$$E_{DS}(\Gamma_G) = 2(3n^2 - 9n + 4) + 2\sqrt{10n^4 - 48n^3 + 81n^2 - 56n + 16}.$$

Proof.

1. By Theorem 3.2 (1), for the odd n, the characteristic polynomial of $DS(\Gamma_G)$ has four eigenvalues, with the first eigenvalue is $\lambda_1 = -2n$ of multiplicity $n-2$, and the second eigenvalue is $\lambda_2 = -2(2n-2)$ of multiplicity $n-1$. The quadratic formula gives the other two eigenvalues, which are $\lambda_3, \lambda_4 = (3n^2 - 6n + 2) \pm \sqrt{10n^4 - 25n^3 + 24n^2 - 12n + 4}$, where one is a positive real number, and the other is negative. Hence, the degree sum energy for Γ_G is
$$E_{DS}(\Gamma_G) = (n-2)|-2n| + (n-1)|-2(2n-2)| + \left|(3n^2 - 6n + 2) \pm \sqrt{10n^4 - 25n^3 + 24n^2 - 12n + 4}\right|$$
$$= 2(3n^2 - 6n + 2) + 2\sqrt{10n^4 - 25n^3 + 24n^2 - 12n + 4}.$$

2. For n is even and following Theorem 3.2 (2), the characteristic polynomial of $DS(\Gamma_G)$ has four eigenvalues, where the first eigenvalue is $\lambda_1 = -2n$ of multiplicity $n-3$, and the second eigenvalue is $\lambda_2 = -2(2n-4)$ of multiplicity $n-1$. The quadratic formula gives the other two eigenvalues, which are $\lambda_3, \lambda_4 = (3n^2 - 9n + 4) \pm \sqrt{10n^4 - 48n^3 + 81n^2 - 56n + 16}$. One is a positive real number for this current case, and the other is negative. Therefore, the degree sum energy for Γ_G is
$$E_{DS}(\Gamma_G) = (n-3)|-2n| + (n-1)|-2(2n-4)| + \left|(3n^2 - 9n + 4) \pm \sqrt{10n^4 - 48n^3 + 81n^2 - 56n + 16}\right|$$
$$= 2(3n^2 - 9n + 4) + 2\sqrt{10n^4 - 48n^3 + 81n^2 - 56n + 16}.$$

DOI: https://doi.org/10.22452/mjs.sp2022no.1.5
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4. Conclusion

This paper has given the general formula of degree sum energy of non-commuting graph for dihedral groups of order $\mathbf{2n}$, $\mathbf{n \geq 3}$. For $\mathbf{n}$ is odd, $E_{DS}(\Gamma_G) = 2(3n^2 - 6n + 2) + 2\sqrt{10n^4 - 25n^3 + 24n^2 - 12n + 4}$, while for $\mathbf{n}$ is even, $E_{DS}(\Gamma_G) = 2(3n^2 - 9n + 4) + 2\sqrt{10n^4 - 48n^3 + 81n^2 - 56n + 16}$.

5. Acknowledgements

This research has been supported by the Ministry of Education (MOE) under the Fundamental Research Grant Scheme (FRGS/1/2019/STG06/UPM/UPM/02/9). We also wish to express our gratitude to Mataram University, Indonesia, for providing partial funding assistance.

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DOI: https://doi.org/10.22452/mjs.sp2022no.1.5
Malaysian Journal of Science 41 (Special Issue 1): 34-39 (September 2022)