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ANALYSIS OF PLANT GROWTH DYNAMICS UNDER THE EFFECT OF TOXICITY: A DELAY DIFFERENTIAL EQUATION MODEL

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Abstract: A mathematical model is designed to examine plant growth under stress in the presence of toxicity with a delay. It is observed that toxic substances change the soil's structure and activity, which has a negative impact on the concentration of nutrients there. The deficiency of soil nutrients and the presence of toxicity are significant elements affecting total biomass. It has been noted that the presence of toxicity changes the physiology and growth of the plant, which ultimately reduces crop growth and production. This adverse effect of toxicity is only seen after an incubation period and is demonstrated by considering the delay in the state variable. Additionally, Hopf bifurcation is observed for the crucial value of the delay parameter. Utilising explicit techniques, the direction and stability of bifurcating periodic solutions are found. Sensitivity analysis is used to determine the sensitivity of solutions of the model when the values of parameters are varied. MATLAB is used for simulation.

Keywords: *Nutrients, plant biomass, toxicity, delay, sensitivity, hopf bifurcation.*

1. Introduction

Plants uptake nutrients from the soil for proper growth as part of the plant-soil interaction process. Macronutrients and micronutrients are the two major types of nutrients found in soil. The macronutrients found in soil, including phosphorus, potassium, nitrogen, calcium, hydrogen and carbon, are advantageous resources that support plant growth. Nickel, zinc and copper are often present in soil at extremely low concentrations and play important roles in plant growth. However, some heavy metals, including chromium, cadmium, lead, mercury, nickel, etc., have a negative impact on soil quality [1] [2]. Excessive levels of heavy metals poison the soil and gradually affect plant growth [3]. Numerous factors, including geological, social, economic and biological ones, contribute to the rise in toxic heavy elements in the soil. Additionally, nutrients play a significant role in discrete plant growth, which has an impact on nonlinear population growth dynamics and, ultimately, on the yield of standing crops [4]. Metals or toxicity cause an imbalance in the soil's nutrition levels. The presence of toxicity affects both the biomass of trees and plants. Thornley was the first to experiment with mathematical modelling in plant physiology by considering various climatic change factors, such as humidity, temperature, rainfall, transpiration, respiration, rate of photosynthesis and guard cells of stomata, among others. However, these models were limited to specific plant species and conditions [5][6]. A mathematical model is proposed to justify the fact that toxic metals have a negative impact on tree biomass [7]. Biomass is negatively affected by the primary and secondary toxicity domains [8]. Also, nutrients have a crucial role in discrete plant growth, influencing the dynamics of nonlinear population increase and, ultimately, the yield of standing crops. Another factor affecting crop yield and crop growth is geographical location [9]. According to a mathematical model, a plant's growth rate is a dynamic process that depends on factors like plant size, decreased growth rate and nutrient mortality rate [10]. Delay was utilised to learn the combined impact of acid and toxic metals on plant populations [11]. The distribution of exponential polynomial roots is explained by Rouches's Theorem (1960). Ruan and Wei (2001) used Rouches' theorem for their consideration of the distribution of exponential polynomial roots [12]. As plant biomass decreases under the influence of toxicity, the variable oscillates for the delay value [13]. Delay was utilised to study the global stability in the collection of non-linear differential equations [14] [15]. It is possible to establish the direction of Hopf bifurcation as well as various numerical simulations using Hassard et al.'s manifold and normal form [16] [17]. The delay differential equations are used to construct the direct and adjoint approaches for sensitivity analysis in bioscience numerical modelling [18]. The sensitivity analysis for a system of nonlinear differential equations with time lags is performed using the 'Direct method' [19]. A generalised method for sensitivity analysis of the delay differential equation is suggested [20] [21]. In relation to the delays, theoretical conclusions for sensitivity are presented. The periodic responses to delay differential equations are studied using a parametric sensitivity analysis [22].

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2. Mathematical Model

Assume that N is the plant nutrient concentration, B is the quantity of plant biomass and T is the amount of toxicity concentration in the plant, all of which serve as three state variables. These are used to model the dynamics of plant growth. The formulation of the model is as follows:

$$\frac{dN}{dt} = N_0 - \alpha_1 N(t-\tau)B - \alpha_2 N - \alpha_3 NT \quad [1]$$

$$\frac{dB}{dt} = rB\left(1-\frac{B}{K}\right) + \beta_1 N(t-\tau)B - \beta_2 B \quad [2]$$

$$\frac{dT}{dt} = T_o - \gamma_1 NT - \gamma_2 T \quad [3]$$

Initially: $N(0) > 0, B(0) > 0, T(0) > 0\ \forall\ t$ and $N(t-\tau) =$ constant for $t \in [-\tau, 0]$.

The parameters are as follows: N_0 represents the fixed amount of nutrients that are available; T_0 denotes the fixed amount of toxicity that are available in soil because of the presence of toxic metals; r represents the growth rate of the plant; K represents the carrying capacity; α_1 is the rate of consumption of nutrients by biomass; α_3 is the rate of decay of nutrients due to its interaction with toxicity; β_1 is the utilisation coefficient of nutrients; and γ_1 is the rate of toxicity decay due to interaction with nutrients. The rates of natural decay for N, B and T are α_2, β_2 and γ_2, respectively. Here, it is assumed that all parameters α_1, α_2, $\alpha_3, \beta_1, \beta_2, \gamma_1, \gamma_{2,} N, B\ and\ T$ are positive.

Boundedness

Lemma 1: Consider the function, $W = N + B + T$

such that, $\frac{dW}{dt} = \frac{dN}{dt} + \frac{dB}{dt} + \frac{dT}{dt}$

Using equations (1)–(3), $\frac{dW(t)}{dt} = No - \alpha_1 NB - \alpha_2 N - \alpha_3 NT + r\left(1-\frac{B}{K}\right) + \beta_1 NB$ - $\beta_2 B + To - \gamma_1 NT - \gamma_2 T$ and $min(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1, \gamma_2) = \delta$ and assuming $N \approx\ N(t-\tau)$ as $t \to \infty$, $\frac{dW(t)}{dt} \le (N_o + T_o)$

By Comparison theorem as $t \to \infty$, $W \le \frac{N_o + T_o}{\delta}$,so $0 \le (N + B + T) \le \frac{N_o + T_o}{\delta}$

Positivity of Solutions

Positivity of system defines that model's solution, with initial data, will eventually be positive for all $\forall\ t$ exceeding some finite value. It is crucial to demonstrate that every solution provided by the equations is a positive solution. Considering equations (1)–(3), where initial condition is $N(0) > 0, B(0) > 0, T(0) > 0\ \forall\ t\ and\ N(t-\tau) = constant\ for\ t\ \in [-\tau, 0]$, the model solution $(N, B, T\ )$ remains positive $\forall$ for all time $t > 0$.

Using equation (3), $\frac{dT}{dt} \ge -\delta(N+1)T$ i.e. $\frac{dT}{dt} \ge -\big((N_o + T_o) + \delta\big)T$,$T \ge c_1 e^{-((N_o+T_o)+\delta)t}$, here c_1 is constant of integration. So, $T > 0\ \forall\ t$. For N and B, the same argument is valid.

Interior Equilibrium Point

A mathematical model under consideration has an equilibrium point that defines a constant solution. We identify the internal equilibrium E^* of the model. For the set of equations (1)–(3), there is only one possible equilibrium at $E^*(N^*, B^*, T^*)$.

$$N^* = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$T^* = \frac{T_o}{\gamma_1 N^* - \gamma_2}$$

$$B^* = \frac{K}{r}(r - \beta_1 N^* - \beta_2)$$

Where a = $\alpha_1 K\beta_1\gamma_1$, b = $\alpha_1\gamma_1 K\ r\ - \alpha_1 K\beta_2\gamma_1 - \alpha_1 K\beta_1\gamma_2 + \alpha_2\gamma_1 r$, c = $\alpha_1 K\ r + \alpha_1 K\beta_2\gamma_1 - \alpha_2\gamma_2 r + \alpha_3 T_o$

Analysis of Hopf bifurcation

This section analyses the dynamical internal equilibrium point behaviour $E^*(N^*, B^*, T^*)$ of model (1)–(3). In relation to the equilibrium E^*, the exponential characteristic equation is provided by

$$\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + (Q_1\lambda^2 + Q_2\lambda + Q_3)e^{-\lambda\tau} = 0 \quad [4],$$

where $P_1 = \alpha_2 + \alpha_3\, T + \frac{r}{k} + \beta_2 + \gamma_1 N + \gamma_2$,

$$P_2 = \frac{r}{k}\alpha_2 + \frac{r}{k}\alpha_3 T + \beta_2\alpha_2 + \beta_2\alpha_3 T + \frac{r}{k}\gamma_1 N + \frac{r}{k}\gamma_2 + \beta_2\gamma_1 N + \beta_2 N_2 + \alpha_2\gamma_1 N + \gamma_1 N\alpha_3 + \gamma_2\alpha_2 + \gamma_2\alpha_3 T,$$

$$P_3 = \frac{r}{k}\alpha_2\gamma_1 N + \frac{r}{k}\ \alpha_2\gamma_2 + \alpha_2\beta_2\gamma_1 N + \alpha_2\beta_2\gamma_2 + \frac{r}{k}\alpha_3\gamma_1 NT + \frac{r}{k}\alpha_3\gamma_2 T + \alpha_3\beta_2\gamma_1 NT + \alpha_3\beta_2\gamma_2 T\text{-}\gamma_1\alpha_3 TN,$$

$$Q_1 = \alpha_1 B,$$

$$Q_2 = \frac{r}{k}\alpha_1 B + \beta_2\alpha_1 B + \gamma_1\alpha_1 N + \gamma_2\alpha_1 B,$$

$$Q_3 = \frac{r}{k}\alpha_1\gamma_1 BN + \frac{r}{k}\alpha_1\gamma_2 B + \alpha_2\beta_2\gamma_1 BN + \alpha_1 B_2\gamma_2 B$$

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Clearly, $P_1, P_2, P_3, Q_1, Q_2, Q_3$ all are positive.
Equation (4) can only be solved if and only if $\lambda = i\omega$ is true.

$$(i\omega)^3 + P_1(i\omega)^2 + P_2(i\omega) + P_3 + (Q_1(i\omega)^2 + Q_2(i\omega) + Q_3)e^{-i\omega\tau} = 0 \quad [5]$$

Separating the real and imaginary parts, we get the following equations:

$$P_3 - P\omega^2 + (Q_3 - Q_1\omega^2)cos\ \omega\tau + Q_2\omega \sin\omega\tau = 0 \quad [6]$$

$$P_2\omega - \omega^3 + Q_2\omega\ cos\ \omega\tau - (Q_3 - Q_1\omega^2)\sin\omega\tau = 0 \quad [7]$$

This further gives:

$$\omega^6 + ({P_1}^2 - {Q_1}^2 - 2P_2)\omega^4 + ({P_2}^2 - {Q_2}^2 + 2Q_1Q_3 - 2P_1P_3)\omega^2 + ({P_3}^2 - {Q_3}^2) = 0 \quad [8]$$

Let

$$u = ({P_1}^2 - {Q_1}^2 - 2P_2), v = ({P_2}^2 - {Q_2}^2 + 2Q_1Q_3 - 2P_1P_3), z = ({P_3}^2 - {Q_3}^2).$$

Let $\omega^2 = x$, then equation (8) becomes $x^3 + ux^2 + vx + z = 0.$ [9]

Claim 1: If $z < 0$, equation (9) has one real positive zero.
Proof: Consider $s(x) = x^3 + ux^2 + vx + z.$
Here, $s(0) = z < 0$ and $\lim_{x\to\infty} s(x) = \infty$. So, $\exists\ z_0 \in (0,\infty)$ such that $s(x_0) = 0.$

Claim 2: If $z \geq 0$, $D = u^2 - 3v \geq 0$ is a necessary condition for the existence of positive real roots in equation (9).
Proof: Since $s(x) = x^3 + ux^2 + vx + z$, therefore $s'(x) = 3x^2 + 2ux + v.$

$s'(x) = 0$ implies $3x^2 + 2ux + v = 0.$ [10]

The roots of equation (10) can be written as $x_{1,2} = \frac{-2u\mp\sqrt{4u^2-12v}}{6} = \frac{-u\mp\sqrt{D}}{3}$ [11]

There are no real roots in equation (10) if $D < 0$. Consequently, the function $s(x)$ is an increasing monotone function in x. Since $k(0) = z \geq 0$, therefore positive real roots cannot exist in equation (9). It has been proven.
Clearly if $D \geq 0$, then $x_1 = \frac{-u+\sqrt{D}}{3}$ is the local minima of $s(x)$. Hence, the following assertion.
Claim 3: If $z \geq 0$, and only if $x_1 > 0$ and $s(x_1) \leq 0$, equation (9) has positive real.
Proof: It is clear that there is enough. There is only one requirement: necessity. If not, assume that $s(x) > 0$ and either $x_1 \leq 0$ or $x_1 > 0$. Consequently, $s(x)$ has no positive real zeros if $x_1 \leq 0$ since $s(x)$ is rising for $x \geq x_1$ and $s(0) = c \geq 0$. Since $x_2 = \frac{-u-\sqrt{D}}{3}$ is the local maximum value if $x_1 > 0$ and $s(x_1) > 0$**,** it follows that $s(x_1) \leq s(x_2)$. Because $s(x)$ lacks positive real roots, $s(0) = c \geq 0$. Proof is now complete.

Lemma 2: Assume that equation (11) defines x_1.
(I) If $z < 0$, at least a positive real zero exists in equation (9).
(II) If $z \geq 0$ and $D = u^2 - 3v < 0$, no positive zeros can be found for equation (9).
(III) If $z \geq 0$, there are positive zeros in equation (9) if $x_1 > 0$ and $s(x_1) \leq 0$.
Proof: Assume that equation (9) has roots that are positive. Suppose it has three constructive roots without losing generality, signified by x_1**,** x_2 , and x_3. Then equation (8) has three positive roots, denoted by $\omega_1 = \sqrt{x_1}$, $\omega_2 = \sqrt{x_2}$, and $\omega_3 = \sqrt{x_3}$.
Using equation (7), $\sin\omega\tau\ = \frac{P_2\omega-\omega^3}{d}$
Which gives $\tau = \frac{1}{\omega}\left[sin^{-1}\left(\frac{P_2\omega-\omega^3}{d}\right) + 2(j-1)\pi\right]$; $j = 1,2,3,-$
Let ${\tau_k}^{(j)} = \frac{1}{\omega_k}\left[sin^{-1}\left(\frac{P_2\omega_k-{\omega_k}^3}{d}\right) + 2(j-1)\pi\right]$; $k = 1,2,3$.; $j = 0,1,2,---$
Then $\mp i\omega_k$ form a pair of equation (8) roots that are entirely imaginary.
Where $\tau = {\tau_k}^{(j)}, k = 1,2,3.; j = 0,1,2,3,--$, $\lim_{j\to\infty} {\tau_k}^{(j)} = \infty\ where\ k = 1,2,3.$
Thus, we define $\tau_0 = {\tau_{k_0}}^{(j_0)} = \min_{1\leq k\leq 3, j\geq 1}\left[{\tau_k}^{(j)}\right],\ \omega_0 = \omega_{k_0},\ x_0 = x_{k_0}$ [12]

Lemma 3: Assume that $P_1 > 0, (P_3 + d) > 0,\ and\ P_1P_2 - (P_3 + d) > 0.$
(I) The real part of very root of equation (4) is negative $\forall\ \tau \geq 0$ if $z \geq 0$ and $D = u^2 - 3v < 0.$
(II) The real part of every root of equation (4) is negative $\forall\ \tau \in [0,\tau_0)$ if $z < 0$ o $z \geq 0$, $x_1 > 0$ and $s(x_1) \leq 0.$
Proof: When $\tau = 0$, equation (4) changes to:

$$\lambda^3 + (P_1 + Q_1)\lambda^2 + (P_2 + Q_2)\lambda + (P_3 + Q_3) = 0. \quad [13]$$

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Using Routh-Hurwitz's criteria, (H1): $if\ (P_3+Q_3)>0, (P_1+Q_1)(P_2+Q_2)-(P_3+Q_3)>0$, then all the roots in equation (4) have negative real parts.
If $z\geq 0$ and $D=u^2-3v<0$, equation (4) does not have any roots with a real part of zero $\forall\ \tau\geq 0$ according to Lemma 2 (II). When $z<0$ or $z\geq 0$, $x>0$ and $s(x_1)\leq 0$, Lemma 2 (I) and (III) implies that when $\tau\neq{\tau_k}^{(j)}, k=1,2,3.; j\geq 1$, Since τ_0 is the smallest value of τ and equation (4) only has imaginary roots, it does not have any real roots with any real parts. The result is obtained using the theorem 1.

Suppose $\lambda(\tau)=\psi(\tau)+i\omega(\tau)$ [14]

being the roots of the equation (4) holds: $\psi(\tau_0)=0,\ \omega(\tau_0)=\omega_0$.
Assume that $s'(x_0)\neq 0$ to ensure that $\mp\omega_0$ are simple and purely imaginary roots of equation (4), as $\tau=\tau_0$ and $\lambda(\tau)$ satisfies the transversality requirement.

Lemma 4: Assume that $x_0={\omega_0}^2$. If $\tau=\tau_0$, then Sign $[\psi'(\tau_0)]$ =Sign $[s'(x_0)]$.
Proof: Differentiating with respect to τ and inserting $\lambda(\tau)$ into equation (4) results in the following:
$\frac{d\lambda}{d\tau}\left[3\lambda^2+2P_1\lambda+Q_2+\left((Q_1\lambda^2+Q_2\lambda+Q_3)(-\tau)+(2Q_1\lambda+Q_2)\right)e^{-\lambda\tau}\right]=\lambda(Q_1\lambda^2+Q_2\lambda+Qn_3)e^{-\lambda\tau}$
Then $\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{(3\lambda^2+2P_1\lambda+P_2)e^{\lambda\tau}}{\lambda(Q_1\lambda^2+Q_2\lambda+Q_3)}+\frac{(2Q_1\lambda+Q_2)}{\lambda(Q_1\lambda^2+Q_2\lambda+Q)}-\frac{\tau}{\lambda}$
From equations (6) - (8):
$\mu'(\tau_0)=Re\left[\frac{(3\lambda^2+2P_1\lambda+P_2)e^{\lambda\tau}}{\lambda(Q_1\lambda^2+Q_2\lambda+Q_3)}\right]+Re\left[\frac{(2Q_1\lambda+Q_2)}{\lambda(Q_1\lambda^2+Q_2\lambda+Q_3)}\right]=\frac{1}{\Delta}[3{\omega_0}^6+2u{\omega_0}^4+v{\omega_0}^2]$
Where $\Delta=[(Q_3-Q\omega^2)^2+(Q_2\omega)^2]$. In this case, $\Delta>0$ and $\omega_0>0$.
Consequently, it is proved that Sign $[\psi'(\tau_0)]$ =Sign $[s'(x_0)]$.

3. Direction Analysis and Stability Analysis of The Hopf Bifurcation Solution

Assuming that $y_1=N-N^*, y_2=B-B^*, y_3=T-T^*$ and time scaling as well as normalising the delay τ , $t\rightarrow\frac{t}{\tau}$, equation (1)–(3) become:

$$\frac{dy_1}{dt}=-\alpha_2y_1-\alpha_1B^*y_1(t-1)-\alpha_1y_1(t-1)y_2-\alpha_3T^*y_1-\alpha_3N^*y_3-\alpha_3y_1y_3 \quad [15]$$

$$\frac{dy_2}{dt}=\frac{r}{k}y_2-\beta_2y_2+\beta_1N^*y_1(t-1)+\beta_1y_1(t-1)y_2 \quad [16]$$

$$\frac{dy_3}{dt}=-\gamma_1T^*y_1-\gamma_1N^*y_3-\gamma_2y_3-\gamma_1y_1y_3 \quad [17]$$

Thus, work can be done in the phase $C=C\left((-1,0),{R_+}^3\right)$. Without loss of generality, denote the critical value τ_j by τ_0. Let $\tau=\tau_0+\mu$, then $\mu=0$ is a Hopf bifurcation value of the system given by equations (15)–(17). Rewrite this system as follows for notational simplicity:

$$y'(t)=L_\mu(y_t)+F(\mu,y_t) \quad [18]$$

Where $y(t)=(y_1(t),y_2(t),y_3(t))^T\in R^3, y_t(\theta)\in C$ is defined by $y_t(\theta)=y_t(t+\theta)$, and
$L_\mu: C\rightarrow R, F: R\times C\rightarrow R$ are provided, respectively by

$$L_\mu\delta=(\tau_0+\mu)\begin{bmatrix}-(\alpha_2+\alpha_3T^*) & 0 & -\alpha_3N^*\\ 0 & -\left(\frac{r}{k}+\beta_2\right) & 0\\ -\gamma_1T^* & 0 & -(\gamma_1N^*+\gamma_2)\end{bmatrix}\begin{bmatrix}\delta_1(0)\\ \delta_2(0)\\ \delta_3(0)\end{bmatrix}+(\tau_0+\mu)\begin{bmatrix}-\alpha_1B^* & 0 & 0\\ -\beta_1B^* & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\delta\,(-1)\\ \delta\,(-2)\\ \delta\,(-3)\end{bmatrix}$$

and $F(\mu,\delta)=(\tau_0+\mu)\begin{bmatrix}F_1\\ F_2\\ F_3\end{bmatrix}$ respectively where $F_1=-\alpha_1\delta_1(-1)\delta_2(0)$,
$F_2=\beta_1\delta_1(-1)\delta_2(0), F_3=-\gamma_1\delta_1(0)\delta_3(0)$,
$\delta(\theta)=\left(\delta_1(\theta),\delta_2(\theta),\delta(\theta)\right)^T\in C\left((-1,0),R\right)$.
According to the Riesz theorem, a function $\eta(\theta,\mu)$ is constrained variation for $\theta\in[-1,0]$, such that $L_\mu\delta=\int_{-1}^{0}d\,\eta(\theta,0)\delta(\theta)$ for $\delta\in C$. Choose in reality.

$$\eta(\theta,\mu)=(\tau_0+\mu)\begin{bmatrix}-(\alpha_2+\alpha_3T^*) & 0 & -\alpha_3N^*\\ 0 & -\left(\frac{r}{k}+\beta_2\right) & 0\\ -\gamma_1T^* & 0 & -(\gamma_1N^*+\gamma_2)\end{bmatrix}\delta(\theta)+\ (\tau_0+\mu)\begin{bmatrix}-\alpha_1B^* & 0 & 0\\ -\beta_1B^* & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\delta(\theta+1)$$

Here, $\delta\in C\left([-1,0],{R_+}^3\right)$ define

$$A(\mu)\delta=\begin{cases}\frac{d\delta(\theta)}{d\theta}, & \theta\in[-1,0)\\ \int_{-1}^{0}d\,\eta(\theta,0)\delta(\theta), & \theta=0.\end{cases}\quad \text{And}\quad R(\mu)\delta=\begin{cases}0, & \theta\in[-1,0)\\ F(\mu,\delta) & \theta=0.\end{cases}$$

The equation (18) then corresponds to:

$$y'(t)=A(\mu)\delta+R(\mu)y_t \quad [19]$$

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For $\psi \in C^1([-1,0], {R_+}^3)$, state

$$A^*\psi(h) = \begin{cases} -\frac{d\psi(h)}{ds}, & h \in [-1,0) \\ \int_{-1}^{0} d\,\eta^T(-t,0)\psi(-t), & h = 0. \end{cases}$$

And bilinear inner product

$$< \psi(h), \delta(\theta) > = \overline{\psi(0)}\delta(0) - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\delta(\xi)\,d\xi \qquad [20]$$

Since A^* and $A = A(0)$ are adjoint operators, and $i\omega_0$ are eigen values of $A(0)$, they are also eigen values of A^*. Assuming that $q(\theta) = q(0)e^{i\omega_0\theta}$ is an eigen vector of $A(0)$ corresponding to the eigen value $i\omega_0$. Then $A(0) = \ i\omega_0\ q(\theta)$. When $\theta = 0$,

$\left[i\omega_0 I - \int_{-1}^{0} d\eta(\theta)e^{i\omega_0\theta}\right]\ q(0) = 0$, the outcome is $q(0) = (1, \sigma_1, \rho_1)^T$

$$\sigma_1 = \frac{(\alpha_1 B^* + (\alpha_2 + \alpha_3 T^*) - i\omega_0)}{\alpha_3 N^*} \ and\ \rho_1 = \frac{\beta_1 B^*\left(\left(\frac{r}{k} + \beta_2\right) - i\omega_0\right)}{\left(\frac{r}{k} + \beta_2\right)^2 + {\omega_0}^2}$$

Similarly, it can be confirmed that $q^*(s) = D(1, \sigma_2, \rho_2)e^{i\omega_0\tau_0 s}$ is the eigen value of A^* that corresponds to $-i\omega_0$, where:

$$\sigma_1 = \frac{(\alpha_1 B^* + (\alpha_2 + \alpha_3 T^*) - i\omega_0)}{\alpha_3 N^*} \ and\ \rho_1 = \frac{\beta_1 B^*\left(\left(\frac{r}{k} + \beta_2\right) - i\omega_0\right)}{\left(\frac{r}{k} + \beta_2\right)^2 + {\omega_0}^2}$$

To ensure $< q^*(s), q(\theta) > = 1$, it is necessary to calculate the value of D.

From equation (22), $< q^*(s), q(\theta) >$

$$= \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2})(1, \sigma_1, \rho_1)^T - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2})e^{-i\omega_0\tau_0(\xi-\theta)}d\eta(\theta)(1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0}\,d\xi$$
$$= \overline{D}\left\{1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} - \int_{-1}^{0}(1, \overline{\sigma_2}, \overline{\rho_2})\ \theta e^{i\omega_0\tau_0\theta}(1, \sigma_1, \rho_1)^T\right\}$$
$$= \overline{D}\{1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} + \tau_0\overline{\sigma_2}W^*(\beta_1\rho_1 - \alpha_1\sigma_1)e^{i\omega_0\tau_0}\}$$

Hence, select $\overline{D} = \frac{1}{(1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} + \tau_0\overline{\sigma_2}B^*(\beta_1\rho_1 - \alpha_1\sigma_1)e^{i\omega_0\tau_0})}$

This ensures that $< q^*(s), q(\theta) > = 1, < q^*(s), \overline{q(\theta)} > = 0$.

The coordinates characterising the centre manifold C_0 at μ = 0 are computed by applying the algorithm described in [16] and using their notations. Assume y_t as a solution of equation (18) at $\mu = 0$. Therefore:

$$z(t) =< q^*(s), y_t(\theta) >,\ \ W(t,\theta) = y_t(\theta) - 2Re(z(t)q(\theta)) \qquad [21]$$

On the centre manifold C_0, $W(t,\theta) = W\left(z(t), \overline{z(t)}, \theta\right)$

Where $W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots$.

Local coordinates for the centre of the manifold C_0 are z and $\overline{z}$ towards q^* and $\overline{q^*}$. Consider that W is real if y_t is real. Only real solutions should be taken into consideration. For the solution $y_t \in C_0$ of equation (20), since $\mu = 0$,

$$z'(t) = i\omega_0\tau_0 z + < \overline{q^*}(\theta), F\left(0, B(z, \overline{z}, \theta) + 2Re\left(z(t)q(\theta)\right)\right) >$$
$$= i\omega_0\tau_0 z + \overline{q^*}(0)\ F\left(0, W(z, \overline{z}, 0) + 2Re\left(z(t)q(\theta)\right)\right)$$
$$\equiv i\omega_0\tau_0 z + \overline{q^*}(0)F_0(z, \overline{z})$$

Rewrite this equation as:

$$z'(t) = i\omega_0\tau_0 z(t) + g(z, \overline{z}) \qquad [22]$$

Where $g(z, \overline{z}) = \overline{q^*}(0)F_0(z, \overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + g_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots$ [23]

As $y_t(\theta) = (y_{1t}, y_{2t}, y_{3t}) = \ W(t,\theta) + z\ q(\theta) + \overline{z}\overline{q(\theta)}$ and $q(0) = (1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0\theta}$, so

$$y_{1t}(0) = z + \overline{z} + {W_{20}}^{(1)}(0)\frac{z^2}{2} + {W_{11}}^{(1)}(0)\ z\overline{z} + {W_{02}}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots,$$
$$y_{2t}(0) = \sigma_1 z + \overline{\sigma_1}\ \overline{z} + {W_{20}}^{(2)}(0)\frac{z^2}{2} + {W_{11}}^{(2)}(0)\ z\overline{z} + {W_{02}}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots,$$
$$y_{3t}(0) = {\rho_1}_1 z + \overline{\rho_{1_1}}\ \overline{z} + {W_{20}}^{(3)}(0)\frac{z^2}{2} + {W_{11}}^{(3)}(0)\ z\overline{z} + {W_{02}}^{(3)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

DOI:https//doi.org/10.22452/mjs.vol43no2.8
Malaysian Journal of Science 43(2): 64-74 (June 2024)

$y_{1t}(-1) = ze^{-i\omega_0\tau_0} + \bar{z}e^{i\omega_0\tau_0} + W_{20}{}^{(1)}(-1)\frac{z^2}{2} + W_{11}{}^{(1)}(-1)\, z\bar{z} + W_{02}{}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots,$

$y_{2t}(-1) = \sigma_1 e^{-i\omega_0\tau_0} z + \overline{\sigma_1}\, e^{i\omega_0\tau_0}\, \bar{z} + W_{20}{}^{(2)}(-1)\frac{z^2}{2} + W_{11}{}^{(2)}(-1)\, z\bar{z} + W_{02}{}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots,$

Thus, comparing coefficients to equation (23) provides:

$g_{20} = \overline{D}(1, \sigma_1, \rho_1) f_{z^2}, g_{02} = \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}) f_{\bar{z}^2},$

$g_{11} = \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}) f_{z\bar{z}}, g_{21} = \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}) f_{z^2\bar{z}}.$

For clarification of g_{21}, computation must be the main focus of $W_{20}(\theta)$ and $W_{11}(\theta)$. From equations (19) and (21):

$$W' = u_t' - z'q - \bar{z}'q = \begin{cases} AW - 2Re\left[\overline{q^*}(0)F_0 q(\theta)\right], & \theta \in [-1,0) \\ AW - 2Re\left[\overline{q^*}(0)F_0 q(0)\right] + F_0, & \theta = 0 \end{cases}$$

Let $W' = AW + H(z, \bar{z}, \theta)$ [24]

Where $H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + H_{21}(\theta)\frac{z^2\bar{z}}{2} + \cdots,$ [25]

As opposed to that, on C_0 at the origin $W' = W_z z' + W_z \bar{z}'$.

The above series is expanded, the coefficients are calculated and the result is:

$[A - 2i\omega_0 I]W_{20}(\theta) = -H_{20}(\theta), AW_{11}(\theta) = -H_{11}(\theta)$ [26]

By equation (22), for $\theta \in [-1,0]$:

$H(z,\bar{z},\theta) = -\overline{q^*}(0)\overline{F_0}q(\theta) - \overline{q^*}(0)\overline{F_0}\,\overline{q}(\theta) = -gq(\theta) - \overline{g}\,\overline{q}(\theta)$

Comparing the coefficients with (23) for $\theta \in [-1,0]$:

$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\overline{q}(\theta), H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\overline{q}(\theta).$

From equation (22), (25) and the definition of A, we obtained:

$W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}}\,\overline{q}(\theta)$

Solving for $W_{20}(\theta)$:

$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{02}}}{3\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta},$

and similarly

$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{11}}}{\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_2,$

where E_1 and E_2 are both three dimensional vectors and can be determined by setting $\theta = 0$ in H. In fact since $H(z,\bar{z},\theta) = -2Re\left[\overline{q^*}(0)F_0 q(0)\right] + F_0$, So

$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\overline{q}(\theta) + F_{z^2},$

$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\overline{q}(\theta) + F_{z\bar{z}}$

Where $F_0 = F_{z^2}\frac{z^2}{2} + F_{z\bar{z}}z\bar{z} + F_{\bar{z}^2}\frac{\bar{z}^2}{2} + \cdots$

Hence combining the definition of A,

$\int_{-1}^{0} d\,\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) + g_{20}q(0) + \overline{g_{02}}\,\overline{q}(0) - F_{z^2}$ and

$\int_{-1}^{0} d\,\eta(\theta)W_{11}(\theta) = g_{11}q(0) - \overline{g_{11}}\,\overline{q}(0) - F_{z\bar{z}}$

Notice that

$\left[i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\right]q(0) = 0$ and $\left[-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right]\overline{q}(0) = 0,$

which implies

$\left[2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta}d\eta(\theta)\right]E_1 = F_{z^2}$ and $-\left[\int_{-1}^{0} d\eta(\theta)\right]E_2 = F_{z\bar{z}}$

Hence,

$$\begin{bmatrix} (2i\omega_0 + \alpha_2 + \alpha_3 T^* + \alpha_1 B^* e^{-2i\omega_0\tau_0}) & 0 & -\alpha_3 N^* \\ -\beta_1 W^* e^{-2i\omega_0\tau_0} & \left(2i\omega_0 + \frac{r}{K} + \beta_2\right) & 0 \\ \gamma_1 T^* & 0 & (2i\omega_0 + \gamma_1 N^* + \gamma_2) \end{bmatrix} E_1 = -2\begin{bmatrix} \alpha_1\sigma_1 e^{-i\omega_0\tau_0\theta} \\ \beta_1\sigma_1 e^{-i\omega_0\tau_0\theta} \\ -\gamma_1\rho_1 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} (\alpha_2 + \alpha_3 T^* + \alpha_1 B^*) & 0 & -\alpha_3 N^* \\ -\beta_1 W^* & \left(\frac{r}{K} + \beta_2\right) & 0 \\ \gamma_1 T^* & 0 & (\gamma_1 N^* + \gamma_2) \end{bmatrix} E_2 = -2\begin{bmatrix} \alpha_1 Re\{\sigma_1\}e^{i\omega_0\tau_0\theta} \\ -\beta_1 Re\{\sigma_1\}e^{i\omega_0\tau_0\theta} \\ -\gamma_1 Re\{\rho_1\} \end{bmatrix}$$

Consequently, the parameters can express g_{21}.

Using the parameters, each g_{ij} can be determined based on the study mentioned above. Consequently, the following values can be calculated:

$C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}}, \beta_2 = 2Re\{C_1(0)\},$

$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0\tau_0}$ [27]

Theorem 2: The value of μ_2 can be determined by the direction of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exists for $\tau > \tau_0$ $(\tau < \tau_0)$. The value of β_2 can determine the stability of bifurcating solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2 < 0$ $(\beta_2 > 0)$. The bifurcating periodic solutions is determined by the value of T_2: the period increases (decreases) if $T_2 > 0$ $(T_2 < 0)$.

DOI:https//doi.org/10.22452/mjs.vol43no2.8
Malaysian Journal of Science 43(2): 64-74 (June 2024)

4. Numerical Stimulation

MATLAB simulation is used to numerically consolidate the analytical findings. The system behaves as follows:
$N_o = 3.17, \alpha_1 = 0.22, \alpha_2 = 0.001, \alpha_3 = 0.0009, r = 1.89, \beta_1 = 0.2, \beta_2 = 0.001, T_o = 2.06, \gamma_1 = 0.06, \gamma_2 = 0.001$
Figure 1 shows that when there is no delay parameter τ, the system is stable. Plant nutrients concentration (*N*), plant biomass (*B*) and toxicity (*T*) show no fluctuation in their natural growth. Figures 2 and 3 show that when the delay parameter τ increased from 0 to 1.24, the system shows limit cycles or perturbation early on but finally stabilises; this is called asymptotically behaviour. Figures 4 and 5 show that when the delay parameter τ crosses the critical value of 1.25, the limit cycle of same period and same direction continue together and Hopf bifurcation occurs.

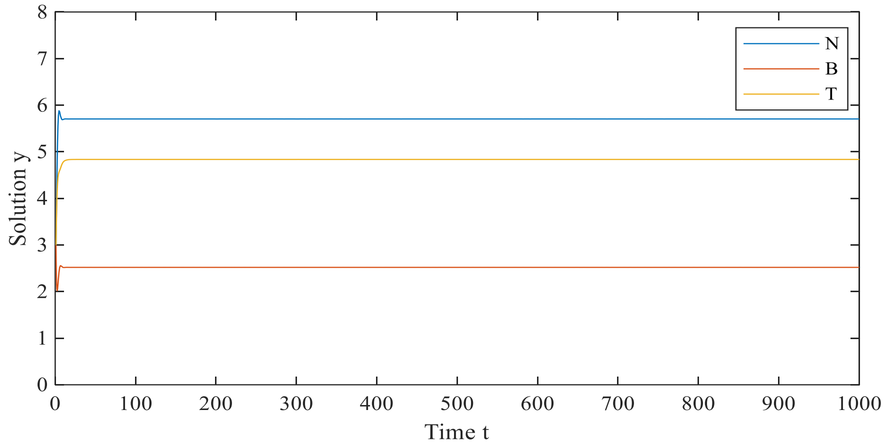

Figure 1. When there is absence of delay, i.e. τ = 0, the system interior equilibrium point E_1 is stable.

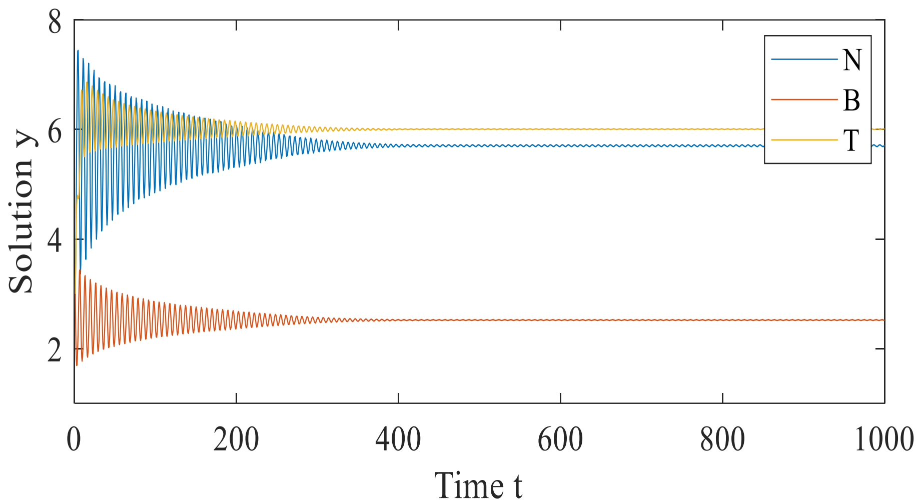

Figure 2. When there is delay, i.e. τ < 1.25, the system interior equilibrium point E_1 is asymptotically stable.

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Malaysian Journal of Science 43(2): 64-74 (June 2024)

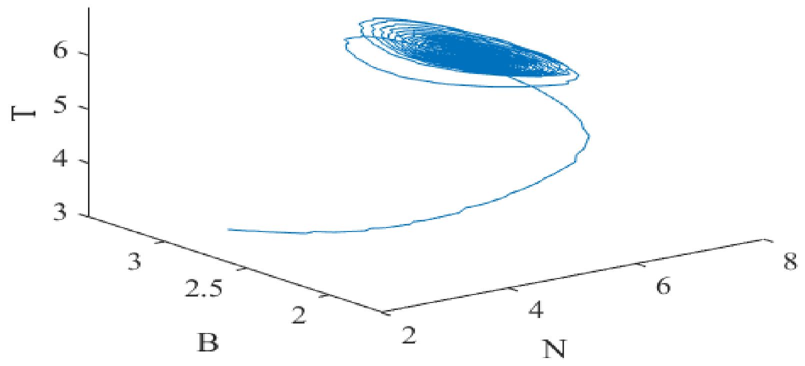

Figure 3. The phase space representation of toxicity (*T*), plant biomass (*B*) and nutrients (*N*) with a delay of $\tau < 1.25$.

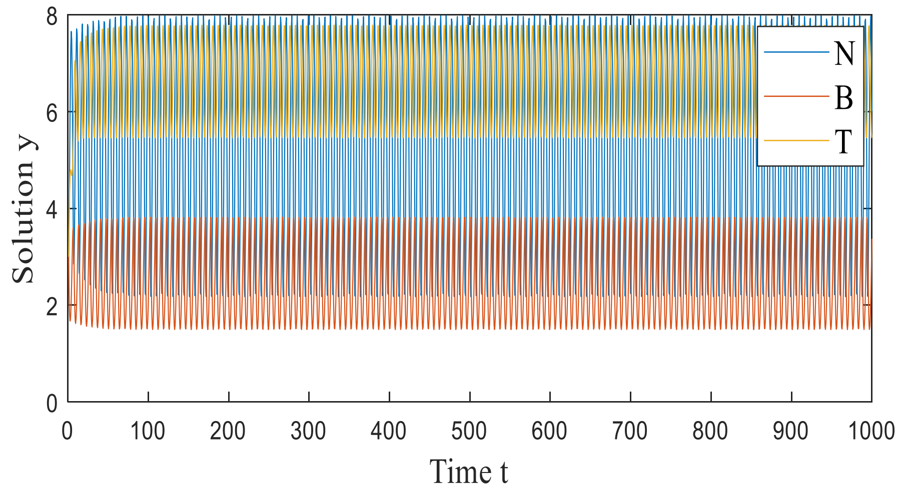

Figure 4. When there is delay, i.e. τ >1.25, the system's interior equilibrium point E_1 loses its stability and shows Hopf bifurcation.

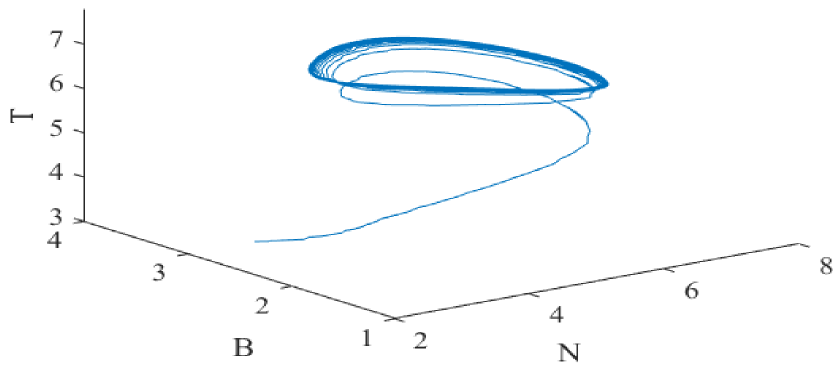

Figure 5. The phase space representation of toxicity (*T*), plant biomass (*B*) and nutrients (*N*) with a delay of $\tau > 1.25$. Asymptotically and orbitally stable is the bifurcating periodic solution.

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Malaysian Journal of Science 43(2): 64-74 (June 2024)

Sensitivity Analysis

The model has constant parameters in this study. To calculate the global sensitivity coefficient, the 'Direct Method' is utilised. For each parameter the partial derivatives of the solution can be found, may be all that is required for sensitivity analysis in this situation if all of the parameters $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1, \gamma_2)$ present in the system (1)–(3) are assumed to be constants. Taking derivative partially of the solution $(N, B \text{ and } T)$ in relation to the β_1, the set of sensitivity equations shown below are produced.

$$\frac{dS_1}{dt} = -\alpha_1 N(t-\tau)S_2 - \alpha_1 BS_1(t-\tau) - \alpha_2 S_1 + \alpha_3 NS_3 - \alpha_3 TS_1 \quad [28]$$

$$\frac{dS_2}{dt} = -\frac{r}{k}S_2 + \beta_1 N(t-\tau)S_2 - \beta_1 BS_1(t-\tau) - \beta_2 S_2 \quad [29]$$

$$\frac{dS_3}{dt} = -\gamma_1 NS_3 - \gamma_1 TS_1 - \gamma_2 S_3 \quad [30]$$

Here $S_1 = \frac{\partial N}{\partial \beta_1}, S_2 = \frac{\partial W}{\partial \beta_1}, S_3 = \frac{\partial M}{\partial \beta_1}$

The nutrient concentration becomes unstable when β_1 = 0.2 and Hopf bifurcation occurs. But when the utilisation coefficient declines from β_1 = 0.2 to β_1 = 0.18, the graph becomes asymptotically stable, and it exhibits stability at β_1 = 0.12 as shown in Figure 6. Similarly, as β_1 drops from β_1 = 0.2 to β_1 = 0.12, as shown in Figures 8 and9, the amount of plant biomass produced and the toxicity decreases respectively.

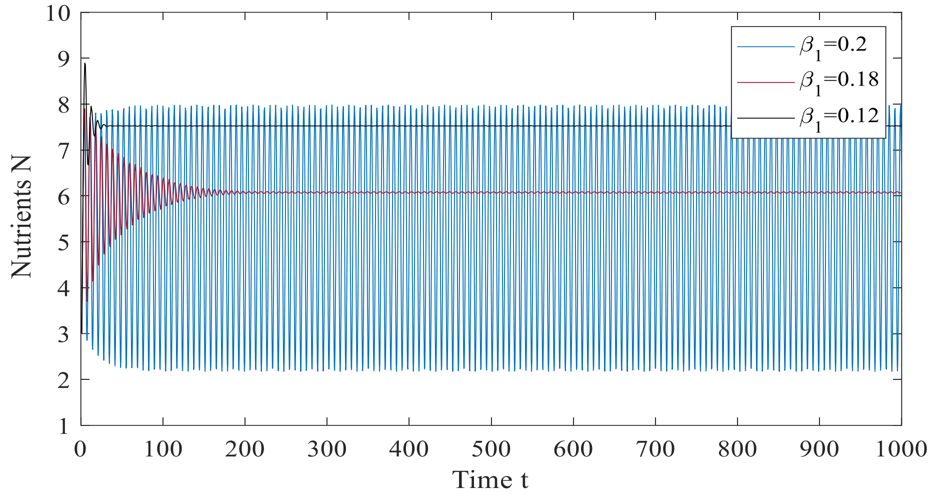

Figure 6. For various values of the utilisation coefficient β_1, a time series graph shows the relationship between small variations in nutrients concentration N.

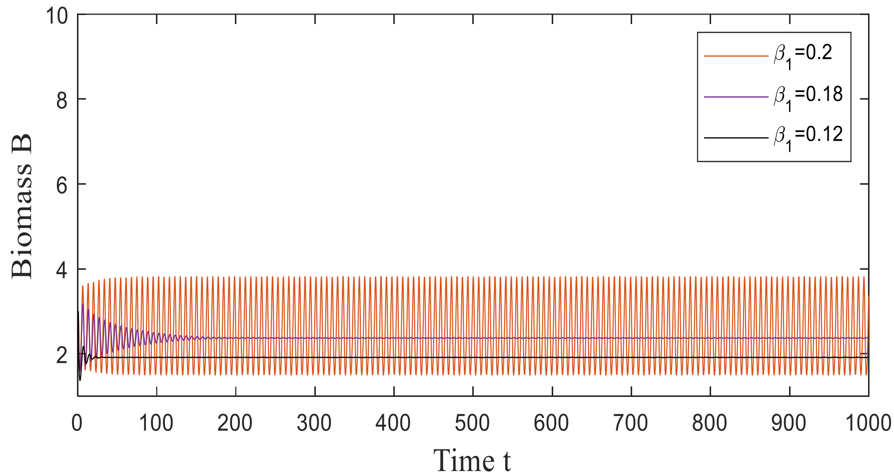

Figure 7. For various values of the utilisation coefficient β_1, time series graph shows the relationship between small variations in biomass B

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Malaysian Journal of Science 43(2): 64-74 (June 2024)

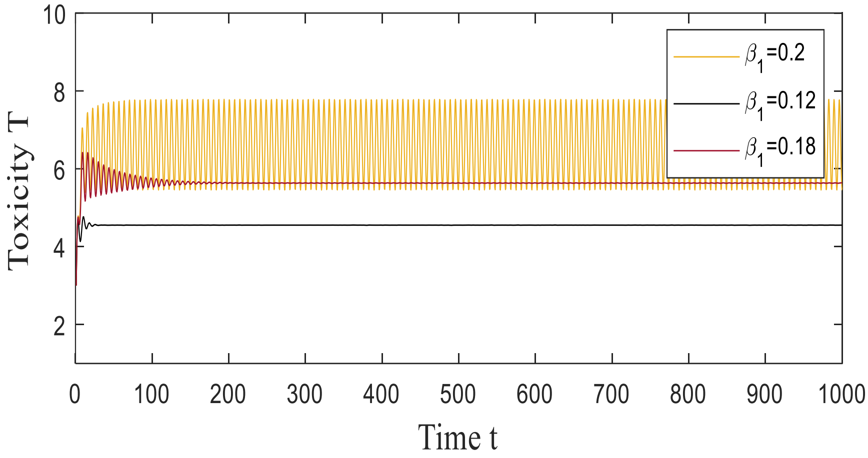

Figure 8. For various values of the utilisation coefficient β_1, a time series graph shows the relationship between small variations in toxic metal T.

5. Conclusion

In this paper, we investigated the impact of delay on the dynamics of plant growth when toxic metals are present. Stable equilibrium, Hopf bifurcation, periodic oscillations, sensitivity analysis, directional analysis and other dynamic phenomena are all seen. Based on some numerical simulations, we draw the conclusion that for some parameter values, the stability and Hopf bifurcation about interior equilibrium E^* can occur. It has been verified that interior equilibrium E^* is stable in the absence of a delay (Figure 1). For a critical value below ($\tau \leq 1.25$) of the parameter delay, the system was asymptotically stable (Figures 2 and 3). The proposed model became unstable and showed oscillations when $\tau \geq 1.25$ (Figures 4 and 5). It was concluded that after taking time lag into account, limit cycles are observed for interior equilibrium points when time delay exceeds a certain critical value. For state variables at the interior equilibrium with respect to the system parameters, sensitivity indices were calculated in the mathematical model (1)–(3). The 'direct method' was used to evaluate sensitivity of state variables by changing the parameter β_1 included in delay differential systems (28)–(30). Analysis of sensitivity demonstrate that the state variable $N, B \; and \; T$ significantly change their rate of oscillations for various values of the parameter β_1. Figures 6–8 depict this phenomenon of sensitivity graphically.

6. References

A. Lacointe, "Carbon allocation among tree organs: A review of basic processes and representation infunctional-structural tree models," 2000.

A. Overman, "A memoir on mathematical models of crop growth and yeild: Effect of geographic location," *Univ. Florida*.

A. R. Watkinson, "Density-dependence in single-species populations of plants," *J. Theor. Biol.*, vol. 83, no. 2, 1980, doi: 10.1016/0022-5193(80)90297-0.

B. Ingalls, M. Mincheva, and M. R. Roussel, "Parametric Sensitivity Analysis of Oscillatory Delay Systems with an Application to Gene Regulation," *Bull. Math. Biol.*, vol. 79, no. 7, 2017, doi: 10.1007/s11538-017-0298-x.

C. Sun, M. Han, and Y. Lin, "Analysis of stability and Hopf bifurcation for a delayed logistic equation," *Chaos, Solitons and Fractals*, vol. 31, no. 3, 2007, doi: 10.1016/j.chaos.2005.10.019.

D. H. Hardy, J. Myers and C. Stokes, *Heavy metals in North Carolina soils : occurrence & significance*. N.C. Department of Agriculture and Consumer Services, Agronomic Division.

D. Kaur and P. Kumar, "Bifurcation induced by delay parameter in plant growth dynamics," *J. Phys. Conf. Ser.*, vol. 2267, 2022.

D. Sharsma, S. Sundar and R. Naresh, "Modelling the effect of toxicant on plant biomass with time delay," *Int. J. non linear Sci.*, vol. 17, pp. 254–267, 2014.

Dipesh and P. Kumar, "Effect of time delay on dynamic of plant competition under allelopathy," *Math. Method Appl. Sci.,*vol. 45, pp.9308–9321, 2022.

F. A. Rihan, "Sensitivity analysis for dynmics systems with time-lags," *J. Comput. Appl. Math.*, vol. 151, no. 2, 2003, doi: 10.1016/S0377-0427(02)00659-3.

G. A. Bocharov and F. A. Rihan, "Numerical modelling in biosciences using delay differential equations," *J. Comput. Appl. Math.*, vol. 125, no. 1–2, 2000, doi: 10.1016/S0377-0427(00)00468-4.

DOI:https//doi.org/10.22452/mjs.vol43no2.8
Malaysian Journal of Science 43(2): 64-74 (June 2024)

G. Deleo, L. Delfuria, and M. Gatto, "The interaction between soil acidity and forest dynamics: A simple-model exhibiting catastrophic behavior," *Theor. Popul. Biol.*, vol. 43, no. 1, 1993, doi: 10.1006/tpbi.1993.1002.

G. Ladas and C. Qian, "Oscillation and global stability in a delay logistic equation," *Dyn. Stab. Syst.*, vol. 9, no. 2, 1994, doi: 10.1080/02681119408806174.

H. T. Banks, D. Robbins, and K. L. Sutton, "Generalized sensitivity analysis for delay differential equations," in *International Series of Numerical Mathematics*, vol. 164, 2013. doi: 10.1007/978-3-0348-0631-2_2.

J. B. Shukla, A. Agarwal, P. Sinha and B. Dubey, "Modelling Effects of Primary and Secondary Toxicant on Renewable Resources.," *Nat. Resour. Model.*, vol. 16, pp. 99–120, 2008.

K. R. Schneider, "Hassard, B. D. / Kazarinoff, N. D. / Wan, Y.-H., Theory and Applications of Hopf Bifurcation. London Mathematical Society Lecture Note Series 41. Cambridge, Cambridge University Press 1981. 320 S., £ 15.00 A P/B. ISBN 0-521-23158-2," *ZAMM - Zeitschrift für Angew. Math. und Mech.*, vol. 62, no. 12, 1982, doi: 10.1002/zamm.19820621221.

O. P. Misra and P. Kalra, "Effect of toxic metal on the structural dry weight of a plant: A Model," *Int. J. Biomath.*, vol. 6, no. 5, 2013, doi: 10.1142/S1793524513500289.

O. P. Misra and P. Kalra, "Modelling Effect of Toxic Metal on the Individual Plant Growth: A Two Compartment Model," *Am. J. Comput. Appl. Math.*, vol. 2, no. 6, 2013, doi: 10.5923/j.ajcam.20120206.06.

S. Trvedi and L. Erdei, "Effects of cadmium and lead on the accumulation of Ca2+ and K+ and on the influx and translocation of K+ in wheat of low and high K+ status," *Physiol. Plant.*, vol. 84, no. 1, 1992, doi: 10.1111/j.1399-3054.1992.tb08770.x.

S. Ruan and J. Wei, "IMA Journal of Mathematics Applications in Medical Biology," *IMA J. Math. Appl. Med. Biol. 1841-52 (2001).*, vol. 18, pp. 41–52, 2001.

S. H. Saker and I. Kubiaczyk, "Oscillation and stability in nonlinear delay differential equations of population dynamics," *Math. Comput. Model.,* vol 35, pp. 298–301*,* 2002.

T. J. H. M, "Mathematical Models in Plant Physiology. A Quantitative Approach to Problems in Plant and Crop Physiology.," *Acad. Press. London*, 1976.

DOI:https//doi.org/10.22452/mjs.vol43no2.8
Malaysian Journal of Science 43(2): 64-74 (June 2024)