A C-F-L condition for the gravity wave and the inertial-gravity waves using an Arakawa *D* lattice

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ABSTRACT The Courant Friedrichs-Levy stability condition for Arakawa's *D* lattice has been derived. The stability condition for this particular grid is less restrictive than the stability condition for grids A and C.

ABSTRAK Keadaan kemantapan Courant Friendrichs-Levy telah ditentukan. Keadaan kemantapan ini adalah kurang terhad berbanding keadaan kemantapan untuk grid A dan C.

INTRODUCTION

Measurements in the ocean are difficult and, quite often, expensive to obtain. Although oceanic data are usually scattered distinctive interpretation of oceanic data is possible. The shallow water model is probably the most powerful tool available to physical oceanographers [1]. In effect, the shallow water model usually provides answers to idealized situations, and also simulates the ocean's response to atmospheric wind forcing. Five different spatial arrangements of variables for the shallow water equations, are generally considered [2]. We will be concern only with Arakawa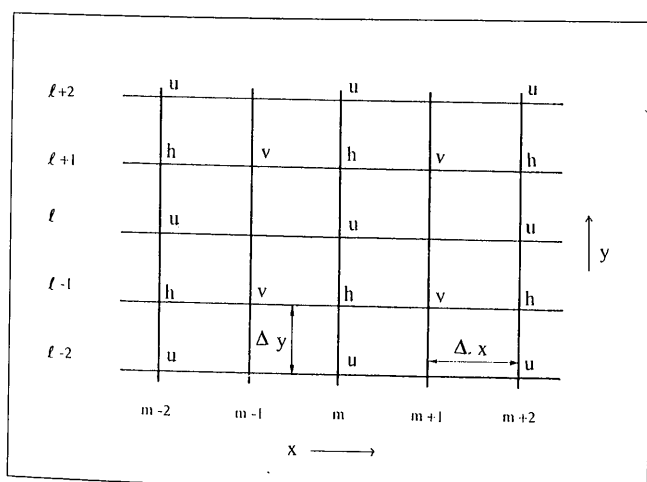's *D* lattice (Fig. 1). Our aim is to gain some understanding about the numerical stability condition for the shallow water equations of Arakawa's D grid [3]. For this purpose, following [4], the stability condition for the one-dimensional gravity wave and the inertial-gravity wave will be considered. For these two types of waves, an analytical study to determine the Courant-Friedrichs-Levy stability condition will be conducted. The technique being used is the one developed by von Neumann [5]. A theoretical analysis, for both the gravity and inertial-gravity waves, shows that the results using lattice *D*, are very similar to the results of the unstaggered grid. [2]. Our analysis shows that the stability condition for the *D* lattice is less restrictive than the stability condition for all other grids. Therefore, there is an advantage for the use of this particular grid. In particular, a larger time step for both a given (spatial) grid resolution and a given mean depth of the ocean could be employed.

THE PROBLEM

The general stability condition of the finite differencing scheme is determined by the general condition:

$$C \, \Delta t / \Delta x \leq O(1),$$

where C is the velocity of the gravity waves, i. e., the fastest traveling waves.

STABILITY ANALYSIS

A. ANALYSIS FOR THE ONE - DIMENSIONAL GRAVITY WAVE

Consider the following set of partial differential equations:

Fig. 1. Representation of variables in Arakawa's *D* lattice.

$$\partial u / \partial t = - g \, \partial h / \partial x$$
$$\partial h / \partial t = - H \, \partial u / \partial x \tag{1}$$

where g represents the earth's gravity; H, the mean sea level depth; u and v, the velocity components in the x (east - west) and y (north - south) directions, respectively; and h, the free surface elevation.

In using a centered in space and time finite difference scheme yields:

$$h_{m, l+1}^{n+1} = h_{m, l+1}^{n-1} - (\gamma H/4) \{(u_{m+2, l+2}^n + u_{m+2, l}^n)$$

$$- (u_{m-2, l+2}^n + u_{m-2, l}^n)\}$$

$$u_{m, l}^{n+1} = u_{m, l}^{n-1} - (\gamma g/4) \{(h_{m+2, l+1}^n + h_{m+2, l-1}^n)$$

$$- (h_{m-2, l+1}^n + u_{m-2, l-1}^n)\} \qquad (2)$$

where γ is equal to $\Delta t / \Delta x$; the superscript n, denotes the time level; the subscripts (m,l), the mesh of discretes grid points in the east-west and north-south directions, respectively, and Δt, the time step increment.

It is convenient to define $C^2 = g H$, θ ($= \mu \Delta x$), and σ ($= \nu \Delta y$), where μ and ν are east-west and north-south wavenumbers, respectively. Assume that $R = (u, v, h)$. Let:

$$R_{m,l}^n = R_n \ \exp(i \mu m \Delta x) \ \exp(i \nu l \Delta y). \qquad (3)$$

Recall that:

$$\sin A = (\exp(i A) - \exp(- i A)) / (2 i),$$
$$\cos A = (\exp(i A) + \exp(- i A)) / (2), \qquad (4)$$

Taking into consideration the set of equations (4), upon substitution of equation (3) into the set of equations (2), it may be obtained:

$$u_{n+1} = u_{n-1} - \gamma g (i \sin 2 \ \theta \cos \sigma) \ h_n$$
$$h_{n+1} = h_{n-1} - \gamma H (i \sin 2 \ \theta \cos \sigma) \ u_n \qquad (5)$$

It is convenient to define an amplification factor, K, such that:

$$R_{n+2} = K R_n \qquad (6)$$

In doing so, equation (5) can then be rewritten as:

$$L_1 \ u_n + L_2 \ h_n = 0$$
$$L_1 \ h_n + L_3 \ u_n = 0 \qquad (7)$$

The operators L_1, L_2 and L_3 are defined as:

$$L_1 = K^{1/2} - K^{-1/2}$$
$$L_2 = i g \gamma \sin 2 \ \theta \cos \sigma$$
$$L_3 = i H \gamma \sin 2 \ \theta \cos \sigma. \qquad (8)$$

The homogeneous set of equations (7) is solved by letting:

$$L_1^2 - L_3 \ L_2 = 0. \qquad (9)$$

A second order equation for K is obtained. Namely:

$$K^2 - 2 (1 - (C \gamma \sin 2 \ \theta \cos \sigma)^2 / 2) \ K + 1 = 0 \qquad (10)$$

The two complex conjugate solutions are:

$$K_+ = F + i (1 - F^2)^{1/2}$$
$$K_- = F - i (1 - F^2)^{1/2}, \qquad (11)$$

where:

$$F = 1 - (C \gamma \sin 2 \ \theta \cos \sigma)^2 / 2. \qquad (12)$$

To have a stable (neutral) condition the absolute value of K should be less (equal) than (to) one. Otherwise, the finite difference scheme is unstable. Multiplying the two solutions, yields:

$$|K|^2 = K_+ \ K_- = 1 \qquad (13)$$

Therefore, if the term under the radical sign, $1 - F^2$, is positive, then the amplification factor will be equal to one. Thus, our stability analysis shows that the chosen finite difference scheme is neutral. This instance will hold true if, and only if:

$$C \gamma \sin 2 \ \theta \cos \sigma \le 2^{1/2} \qquad (14)$$

Recall the definition of γ. Due to the fact that the absolute value of sines and cosines are less or equal than unity, it is inferred:

$$C \Delta t / \Delta x \le 2^{1/2} \qquad (15)$$

which is the typical C-F-L condition for computational stability. This stability condition, for the D lattice, is less restrictive than the stability condition for all other lattices.

B. ANALYSIS FOR THE INERTIAL-GRAVITY WAVE

Consider the following set of partial differential equations:

$$\partial u / \partial t = f v - g \partial h / \partial x$$
$$\partial v / \partial t = - f u$$
$$\partial h / \partial t = - H \partial u / \partial x , \qquad (16)$$

where f, the Coriolis parameter, has a typical value of less (or equal) than 10^{-5} sec^{-1} at Malaysian latitudes.

The same finite differencing scheme as for the gravity wave, and taking into consideration the set of equations (4), yields:

$$u_{n+1} = u_{n-1} + 0.5 \Delta t f \cos \theta \cos \sigma v_n - i g \gamma \sin 2 \theta \cos \sigma h_n$$

$$v_{n+1} = v_{n-1} - 0.5 \Delta t f \cos \theta \cos \sigma u_n$$

$$h_{n+1} = h_{n-1} - i H \gamma \sin 2 \theta \cos \sigma h_n \qquad (17)$$

The set of equations (17) may then be rewritten as:

$$L_1 u_n - L_4 v_n + L_2 h_n = 0$$
$$L_1 v_n + L_4 u_n = 0$$
$$L_1 h_n + L_3 u_n = 0, \qquad (18)$$

where $L_4 = \Delta t f \cos \theta \cos \sigma / 2$.

Following the same procedure, as in the previous case, yields:

$$K = (1 - (\Delta t f \cos \theta \cos \sigma/2.)^2$$
$$- (C \gamma \sin 2 \theta \cos \sigma)^2 /2)$$
$$\pm i \{ 1 - [1 - (\Delta t f \cos \theta \cos \sigma/2.)^2$$
$$- (C \gamma \sin 2 \theta \cos \sigma)^2 /2] \}^{1/2} \qquad (19)$$

If the term under the radical sign is positive, the stability analysis shows that the absolute value of the amplification factor is equal to one. Thus, we will have a neutral stability condition. This will require that:

$$(\Delta t f \cos \theta \cos \sigma./2)^2$$
$$+ (C \gamma \sin 2 \theta \cos \sigma)^2 / 2 \leq 1 \qquad (20)$$

Due to the fact that the absolute value of sines and cosines are less or equal to unity, yields:

$$C \Delta t / \Delta x \leq \{ 2 [1 - (f \Delta t / 2)^2] \}^{1/2} \qquad (21)$$

The same result as in the previous case is obtained. Namely, that the stability condition, for the inertial-gravity waves, for the D grid, is less restrictive than for all other lattices.

CONCLUSION

Our analysis shows that the stability condition for the D grid for the fastest traveling wave, is double the value than for the other four grids. A larger time step, for a given (spatial) resolution and a given mean depth of the ocean, may be used.

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