INTEGRATION FOR SPECIAL THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS USING IMPROVED RUNGE-KUTTA DIRECT METHOD

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ABSTRACT   In this paper, we derive an explicit four stage fifth-order Improved Runge-Kutta (IRKD) method for numerical integration of special third-order ordinary differential equation. The method proposed here is two-step in nature and require less number of stages per step compared with the existing Runge-Kutta (RK) method. The stability polynomial of the IRKD method is presented. Numerical results are given to illustrate the efficiency of the proposed method compared to the RK method and direct Runge-Kutta (RKD) method for solving special third-order ordinary differential equations.

(**Keywords**: Special third-order ordinary differential equations, Runge-Kutta method, IRKD method, Stability Ploynomial)

INTRODUCTION

In this article, we are concerned with the special third-order ordinary differential equations (ODEs), which is defined as follows:

$$y'''(x) = f(x, y) \qquad (1)$$

with initial conditions

$$y(x_0) = \eta_0, \ y'(x_0) = \eta'_0, \ y''(x_0)) = \eta''_0.$$

where $f : R \times R \to R$ is a continuous valued function which does not contain the first derivative $y'(x)$ and second derivative $y''(x)$. This type of problems often arises in a variety of applied sciences and engineering such as physics, molecular dynamics, the motion of rocket, biology and chemistry. Traditionally, most researchers solve the third-order ODE by transforming it to a system of the first order ordinary differential equations ODEs, then solve it using a suitable numerical method (see [10, 11, 12, 13]). However, Several researchers (see [3, 4]) observed the drawback of this technique whereby it wasted a lot of computing time and human effort. Therefore, direct integration methods have attracted significant attention from several authors for solving third-order ODEs. For instance, Awoyemi [1] constructed the P-stable multistep collocation method to solve directly third-order ODEs using constant step size. Suleiman [8] presented a direct integration method to solve higher order ODEs using variable step size codes. Awoyemi and Idowu [2] derived the hybrid collocation method for solving third order ordinary differential equations. Waeleh et al. [5] presented the two point block method to to solve third-order ODEs directly. Majid et al. [7] introduced a block implicit method for directly solving third-order ODEs using variable step size codes. This paper primarily aims to derive a new improved Runge-Kutta method, namely IRKD method to solve special third-order ODEs directly.

THE GENERAL FORM OF IRKD METHOD

Motivated by the technique in Rabiei [9] for obtaining the algebraic order equations for Runge-Kutta Nyström method, we derive the algebraic order conditions for the IRKD method for solving special third-order ODEs directly. Consequently, the general $s -$stages IRKD method for solving (1) can be defined as follows:

$$y_{n+1} = y_n + \frac{3}{2}hy'_n - \frac{1}{2}hy'_{n-1} + \frac{5}{12}h^2y''_n$$

$$- \frac{5}{12}h^2y''_{n-1} + h^3\sum_{i=2}^{s}b''_i(k_i - k_{-i}), \qquad (2)$$

$$y'_{n+1} = y'_n + \frac{3}{2}hy''_n - \frac{1}{2}hy''_{n-1} + h^2\sum_{i=2}^{s}b'_i(k_i - k_{-i}), \qquad (3)$$

$$y''_{n+1} = y''_n + h(b_{-1}k_{-1} + b_1k_1 + \sum_{i=2}^{s}b_i(k_i - k_{-i})), \qquad (4)$$

where

$$k_1 = f(x_n, y_n), \qquad (5)$$

$$k_{-1} = f(x_{n-1}, y_{n-1}), \qquad (6)$$

$$k_i = f(x_n + c_ih, y_n + hc_iy'_n + \frac{1}{2}h^2c_i^2y''_n + $$
$$+ h^3\sum_{j=1}^{s}a_{ij}k_j) \qquad (7)$$

$$k_{-i} = f(x_{n-1} + c_ih, y_{n-1} + hc_i + \frac{1}{2}h^2c_i^2y''_{n-1} + h^3\sum_{j=1}^{s}a_{ij}k_{-j}). \qquad (8)$$

All coefficients b_i, b'_i, b''_i, a_{ij}, and c_i of the IRKD method are supposed to be real and for $i = 2,,,...,s$ and $j = 1,2,...,s$.

The IRKD method expression (2)-(8) is extended using Taylor series expansion to obtain the coefficients of the IRKD method. After doing some algebraic simplifications, this expansion is equated to the true solution which is presented by the Taylor series expansion. As a result, the system of nonlinear algebraic equations is obtained, which is denoted as order conditions. The algebraic and numerical calculations are carried out using the Maple package. The IRKD method can be represented in Butcher notation as follows:

$$
\begin{array}{c|c}
c & A \\
\hline
b_{-1} & b^T \\
& b'^T \\
& b''^T
\end{array}
$$

DERIVATION OF FIFTH-ORDER IRKD METHOD WITH FOUR STAGE

The general form of four-stage fifth-order IRKD method has the following form

$$y_{n+1} = y_n + \frac{3}{2}hy'_n - \frac{1}{2}y'_{n-1} + \frac{5}{12}h^2y''_n - \frac{5}{12}h^2y''_{n-1}$$

$$+ h^3(b''_2(k_2 - k_{-2}) + b''_3(k_3 - k_{-3})$$

$$+ b''_4(k_4 - k_{-4})) \qquad (9)$$

$$y'_{n+1} = y'_n + \frac{3}{2}hy''_n - \frac{1}{2}hy''_{n-1} + h^2(b'_2(k_2 - k_{-2})$$

$$+ b'_3(k_3 - k_{-3}) + b'_4(k_4 - k_{-4})), \qquad (10)$$

$$y''_{n+1} = y''_n + h(b_{-1}k_{-1} + b_1k_1 + b_2(k_2 - k_{-2})$$

$$+ b_3(k_3 - k_{-3}) + b_4(k_4 - k_{-4})), \qquad (11)$$

where

$$k_1 = f(x_n, y_n),$$

$$k_{-1} = f(x_{n-1}, y_{n-1}),$$

$$k_2 = f(x_n + c_2h, y_n + hc_2y'_n + \frac{1}{2}h^2c_2^2y''_n + h^3(a_{21}k_1) \qquad (12)$$

$$k_{-2} = f(x_{n-1} + c_2h, y_{n-1} + hc_2y'_{n-1} + \frac{1}{2}h^2c_2^2y''_{n-1} + h^3(a_{21}k_{-1}) \qquad (13)$$

$$k_3 = f(x_n + c_3h, y_n + hc_3y'_n + \frac{1}{2}h^2c_3^2y''_n + h^3(a_{31}k_1 + a_{32}k_2)), \qquad (14)$$

$$k_{-3} = f(x_{n-1} + c_3h, y_{n-1} + hc_3y'_{n-1} + \frac{1}{2}h^2c_3^2y''_{n-1} + h^3(a_{31}k_{-1} + a_{32}k_{-2})), \qquad (15)$$

$$k_4 = f(x_n + c_4h, y_n + hc_4y'_n + \frac{1}{2}h^2c_4^2y''_n + h^3(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)), \qquad (16)$$

$$k_{-4} = f(x_{n-1} + c_4h, y_{n-1} + hc_4y'_{n-1} + \frac{1}{2}h^2c_4^2y''_{n-1} + h^3(a_{41}k_{-1} + a_{42}k_{-2} + a_{43}k_{-3})). \qquad (17)$$

By applying the Taylor series expansion on IRKD method (9) - (17) we obtain the order conditions up to order six as follows:

The order conditions for y:

order 4:

$$\sum_{i=2}^{4}b''_i = \frac{1}{6}, \qquad (18)$$

order 5:

$$\sum_{i=2}^{4} b_i'' c_i = \frac{31}{720}, \tag{19}$$

order 6:

$$\sum_{i=2}^{4} b_i'' c_i^2 = \frac{1}{60}. \tag{20}$$

The order conditions for y':

order 3:

$$\sum_{i=2}^{4} b_i' = \frac{5}{12}, \tag{21}$$

order 4:

$$\sum_{i=2}^{4} b_i' c_i = \frac{1}{6}, \tag{22}$$

order 5:

$$\sum_{i=2}^{4} b_i' c_i^2 = \frac{31}{360}, \tag{23}$$

order 6:

$$\sum_{i=2}^{4} b_i' c_i^3 = \frac{1}{20}, \tag{24}$$

$$\sum_{i=2}^{4} b_i' a_{ij} = \frac{1}{120}. \tag{25}$$

The order conditions for y'':

order 1:

$$b_1 - b_{-1} = 1, \tag{26}$$

order 2:

$$b_{-1} + \sum_{i=2}^{4} b_i = \frac{1}{2}, \tag{27}$$

order 3:

$$\sum_{i=2}^{4} b_i c_i = \frac{5}{12}, \tag{28}$$

order 4:

$$\sum_{i=2}^{4} b_i c_i^2 = \frac{1}{3}, \tag{29}$$

order 5:

$$\sum_{i=2}^{4} b_i c_i^3 = \frac{31}{120}, \tag{30}$$

$$\sum_{i=2}^{4} b_i a_{ij} = \frac{31}{720}, \tag{31}$$

order 6:

$$\sum_{i=2}^{4} b_i c_i^4 = \frac{1}{5}, \tag{32}$$

$$\sum_{i=2}^{4} b_i a_{ij} c_j = \frac{1}{120}, \quad \sum_{i=2}^{4} b_i c_i a_{ij} = \frac{1}{30}. \tag{33}$$

To construct a four-stage fifth-order IRKD method, the order conditions up to order five (18)-(19), (21)-(23) and (26)-(31), which include 11 nonlinear equations with 20 unknowns need to be solved. Therefore, the resulting system has nine free parameters; setting $c_2 = \frac{1}{12}$, $c_3 = \frac{2}{9}$, $c_4 = \frac{2}{3}$, $a_{32} = \frac{1}{32}$ and $b_2'' = \frac{1}{20}$, solving the system simultaneously yields a solution as follows:

$$b_1 = -\frac{27}{40}, \ b_2 = \frac{88}{25}, \ b_3 = -\frac{459}{200}, \ b_4 = \frac{19}{20}, \ b_{-1} = -\frac{67}{40},$$

$$b_2' = -\frac{2}{525}, \ b_3' = \frac{51}{200}, \ b_4' = \frac{139}{840}, \ b_3'' = \frac{7}{80}, \ b_4'' = \frac{7}{240},$$

$$a_{42} = -\frac{352}{95} a_{21} + \frac{459}{190} a_{31} + \frac{6611}{54720} - a_{41} - a_{43}.$$

As a results, we have four free parameters a_{21}, a_{31}, a_{41} and a_{43} which can be chosen by minimizing the error equations. According to Dormand et al. [14], the free parameters can be selected by minimizing the global error of the sixth-order conditions. The approach is performed as follows:

Firstly: the error equations of y, y' and y'' is determined respectively as follows:

$$\left\| \tau''^{(6)} \right\|_2 = \sqrt{\sum_{i=1}^{n_p''+1} \left(\tau_i''^{(6)}\right)^2}, \tag{34}$$

$$\left\| \tau'^{(6)} \right\|_2 = \sqrt{\sum_{i=1}^{n_p'+1} \left(\tau_i'^{(6)}\right)^2}, \tag{35}$$

$$\left\| \tau^{(6)} \right\|_2 = \sqrt{\sum_{i=1}^{n_p+1} \left(\tau_i^{(6)}\right)^2}. \tag{36}$$

Secondly: The global error norm is determined as follows:

$$\left\| \tau_g^{(6)} \right\|_2 =$$
$$\sqrt{\sum_{i=1}^{n_p''+1} \left(\tau_i''^{(6)}\right)^2 + \sum_{i=1}^{n_p'+1} \left(\tau_i'^{(6)}\right)^2 + \sum_{i=1}^{n_p+1} \left(\tau_i^{(6)}\right)^2} \tag{37}$$

Finally: minimize the global error in equation (37) with respect to the free parameters a_{21}, a_{31}, a_{41} and a_{43},

Therefore, we obtain the free parameters as follows:

$$a_{21} = -\frac{367}{120344}, \quad a_{31} = -\frac{13703}{417136}, \quad a_{41} = \frac{20457}{17918}$$

and $a_{43} = \frac{34752}{45617}$. Which leads to $a_{42} = -\frac{41509}{22428}$ and then the error coefficients for y, y' and y'' are computed respectively as follows:

$$\left\| \tau''^{(6)} \right\|_2 = 9.645061728 \times 10^{-4},$$

$$\left\| \tau'^{(6)} \right\|_2 = 1.826131687 \times 10^{-3},$$

$$\left\| \tau^{(6)} \right\|_2 = 1.777263374 \times 10^{-2},$$

and the global error norm is

$$\left\| \tau_g^{(6)} \right\|_2 = 1.789222008 \times 10^{-2}.$$

where $\tau''^{(6)}$, $\tau'^{(6)}$ and $\tau^{(6)}$ are the error equations of sixth-order conditions for y, y' and y'' respectively. Finally, the fifth-order IRKD method with four-stage are written in Butcher tableau and denoted as IRKD5 method as follows:

Table 1. Butcher Tableau of IRKD5 Method

$\frac{1}{12}$	$-\frac{367}{120344}$			
$\frac{2}{9}$	$-\frac{13703}{417136}$	$\frac{1}{32}$		
$\frac{2}{3}$	$\frac{20457}{17918}$	$-\frac{41509}{22428}$	$\frac{34752}{45617}$	
$-\frac{67}{40}$	$\frac{27}{40}$	$\frac{88}{25}$	$-\frac{459}{200}$	$\frac{19}{20}$
		$-\frac{2}{525}$	$\frac{51}{200}$	$\frac{139}{840}$
		$\frac{1}{20}$	$\frac{7}{80}$	$\frac{7}{240}$

STABILITY POLYNOMIAL OF IRKD METHOD

To obtain the stability polynomial of the IRKD method, we can rewrite the IRKD method as follows:

$$y_{n+1} = y_n + \frac{3}{2}hy'_n - \frac{1}{2}hy'_{n-1} + \frac{5}{12}h^2 y''_n - \frac{5}{12}h^2 y''_{n-1}$$

$$+ h^3 \sum_{i=2}^{s} b''_i \left(f(x_n + c_i h, Y_i) - \right.$$

$$\left. f(x_{n-1} + c_i h, Y_{-i}) \right), \qquad (38)$$

$$y'_{n+1} = y'_n + \frac{3}{2}hy''_n - \frac{1}{2}hy''_{n-1}$$

$$+ h^2 \sum_{i=2}^{s} b'_i \left(f(x_n + c_i h, Y_i) - \right.$$

$$\left. f(x_{n-1} + c_i h, Y_{-i}) \right), \qquad (39)$$

$$y''_{n+1} = y''_n + h(b_{-1}Y_{-1} + b_1 Y_1$$

$$+ h^2 \sum_{i=2}^{s} b_i \left(f(x_n + c_i h, Y_i) - \right.$$

$$\left. f(x_{n-1} + c_i h, Y_{-i}) \right), \qquad (40)$$

where

$$Y_1 = y_n, \qquad (41)$$

$$Y_{-1} = y_{n-1}, \qquad (42)$$

$$Y_i = y_n + hc_i y'_n + \frac{1}{2}h^2 c_i^2 y''_n$$

$$+ h^3 \sum_{j=1}^{i-1} a_{ij} \left(f(x_n + c_i h, Y_i) \right) \qquad (43)$$

$$Y_{-i} = y_{n-1} + hc_i y'_{n-1} + \frac{1}{2}h^2 c_i^2 y''_{n-1}$$

$$+ h^3 \sum_{j=1}^{i-1} a_{ij} \left(f(x_{n-1} + c_i h, Y_{-i}) \right). \qquad (44)$$

for $i = 2,3, \dots, s$.

We consider the following test equation

$$y''' = -\lambda^3 y. \qquad (45)$$

Now, applying the IRKD method (38)-(44) to the test equation (45), which leads to

$$y_{n+1} = y_n + \frac{3}{2}hy'_n - \frac{1}{2}hy'_{n-1} + \frac{5}{12}h^2 y''_n - \frac{5}{12}h^2 y''_{n-1}$$

$$+ h^3 \sum_{i=2}^{s} -\lambda^3 b''_i (Y_i - Y_{-i}), \qquad (46)$$

$$y'_{n+1} = y'_n + \frac{3}{2}hy''_n - \frac{1}{2}hy''_{n-1}$$

$$+ h^2 \sum_{i=2}^{s} -\lambda^3 b'_i (Y_i - Y_{-i}), \qquad (47)$$

$$y''_{n+1} = y''_n + h((-\lambda^3 b_{-1})Y_{-1} + (-\lambda^3 b_1)Y_1$$

$$+ \sum_{i=2}^{s} -\lambda^3 b_i (Y_i - Y_{-i})), \qquad (48)$$

where

$$Y_i = y_n + hc_i y'_n + \frac{1}{2}h^2 c_i^2 y''_n + h^3 \sum_{j=1}^{i-1} -\lambda^3 a_{ij} Y_i, \qquad (49)$$

$$Y_{-i} = y_{n-1} + hc_i y'_{n-1} + \frac{1}{2}h^2 c_i^2 y''_{n-1}$$

$$+ h^3 \sum_{j=1}^{i-1} -\lambda^3 a_{ij} Y_{-i}. \qquad (50)$$

Multiplying equations (47) by h and (48) by h^2, we get

$$hy'_{n+1} = hy'_n + \tfrac{3}{2}h^2 y''_n - \tfrac{1}{2}h^2 y''_{n-1} + (-\lambda^3 h^3)\sum_{i=2}^{s} b'_i (Y_i - Y_{-i}), \tag{51}$$

$$h^2 y''_{n+1} = h^2 y''_n + (-\lambda^3 h^3)\left(b_{-1}Y_{-1} + b_1 Y_1 + \sum_{i=2}^{s} b_i (Y_i - Y_{-i})\right) \tag{52}$$

The equations (46), (51) and (52) can be represented in the following matrix format

$$\begin{bmatrix} y_{n+1} \\ y_n \\ hy'_{n+1} \\ hy'_n \\ h^2 y''_{n+1} \\ h^2 y''_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \tfrac{3}{2} & -\tfrac{1}{2} & \tfrac{5}{12} & -\tfrac{5}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \tfrac{3}{2} & -\tfrac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \\ h^2 y''_n \\ h^2 y''_{n-1} \end{bmatrix} + (-\lambda^3 h^3)\begin{bmatrix} 0 & 0 & b''_2 & b''_3 & \cdots & b''_s \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b'_2 & b'_3 & \cdots & b'_s \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ b_{-1} & b_1 & b_2 & b_3 & \cdots & b_s \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} - \begin{bmatrix} Y_{-1} \\ 0 \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix}\right) \tag{53}$$

Also, the matrix format of equations (49) and (50) is defined as

$$\begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & c_2 & 0 & \tfrac{1}{2}c_2^2 & 0 \\ 1 & 0 & c_3 & 0 & \tfrac{1}{2}c_3^2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & c_s & 0 & \tfrac{1}{2}c_s^2 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \\ h^2 y''_n \\ h^2 y''_{n-1} \end{bmatrix} + (-\lambda^3 h^3)\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{21} & 0 & \cdots & 0 & 0 \\ a_{31} & a_{31} & a_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{s1} & a_{s1} & a_{s2} & \cdots & a_{ss-1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} \tag{54}$$

and

$$\begin{bmatrix} 0 \\ Y_{-1} \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & c_2 & 0 & \tfrac{1}{2}c_2^2 \\ 0 & 1 & 0 & c_3 & 0 & \tfrac{1}{2}c_3^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & c_s & 0 & \tfrac{1}{2}c_s^2 \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \\ h^2 y''_n \\ h^2 y''_{n-1} \end{bmatrix} + (-\lambda^3 h^3)\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{21} & 0 & \cdots & 0 & 0 \\ a_{31} & a_{31} & a_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{s1} & a_{s1} & a_{s2} & \cdots & a_{ss-1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Y_{-1} \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix} \tag{55}$$

After carrying out some algebraic manipulation, we obtain the final matrix as follows

$$u_{n+1} = Pu_n + HBY, \tag{56}$$

$$Y = Cu_n + HAY. \tag{57}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{21} & 0 & \cdots & 0 & 0 \\ a_{31} & a_{31} & a_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{s1} & a_{s1} & a_{s2} & \cdots & a_{ss-1} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & b''_2 & b''_3 & \cdots & b''_s \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b'_2 & b'_3 & \cdots & b'_s \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ b_{-1} & b_1 & b_2 & b_3 & \cdots & b_s \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$P = \begin{bmatrix} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{5}{12} & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Simplifying (57) we have,

$$Y = (I - HA)^{-1}Cu_n, \qquad (58)$$

where I is the identity matrix. Finally, substituting (58) into (56), we get

$$u_{n+1} = (P + HB((I - HA)^{-1}C))u_n. \qquad (59)$$

Therefore, equation (59) can be written as

$$u_{n+1} = W(H)u_n, \qquad (60)$$

where

$$W(H) = \begin{bmatrix} 1 + Hb''^T P^{-1}C_1 & Hb''^T P^{-1}C_2 & \frac{3}{2} + Hb''^T P^{-1}C_3 & -\frac{1}{2} + Hb''^T P^{-1}C_4 & \frac{5}{12} + Hb''^T P^{-1}C_5 & -\frac{5}{12} + Hb''^T P^{-1}C_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ Hb'^T P^{-1}C_1 & Hb'^T P^{-1}C_2 & 1 + Hb'^T P^{-1}C_3 & Hb'^T P^{-1}C_4 & \frac{3}{2} + Hb'^T P^{-1}C_5 & -\frac{1}{2} + Hb'^T P^{-1}C_6 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ Hb^T P^{-1}C_1 & Hb^T P^{-1}C_2 & Hb^T P^{-1}C_3 & Hb^T P^{-1}C_4 & 1 + Hb^T P^{-1}C_5 & Hb^T P^{-1}C_6 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (61)$$

Also where

$$P^{-1} = (I - HA)^{-1}, \quad b'' = [0 \quad 0 \quad b_2'' \quad ... \quad b_s'']^T, \quad b' = [0 \quad 0 \quad b_2' \quad ... \quad b_s']^T, \quad b = [0 \quad 0 \quad b_2 \quad ... \quad b_s]^T,$$

$$C_1 = [0 \quad 1 \quad 1 \quad ... \quad 1]^T, \quad C_2 = [-1 \quad 0 \quad -1 \quad ... \quad -1]^T, \quad C_3 = [0 \quad 0 \quad c_2 \quad ... \quad c_s]^T,$$

$$C_4 = [0 \quad 0 \quad -c_2 \quad ... \quad -c_s]^T, \quad C_5 = \left[0 \quad 0 \quad \frac{1}{2}c_2^2 \quad ... \quad \frac{1}{2}c_s^2\right]^T, \quad C_6 = \left[0 \quad 0 \quad -\frac{1}{2}c_2^2 \quad ... \quad -\frac{1}{2}c_s^2\right]^T.$$

Thus, the stability polynomial associated with the IRKD method is defined by

$$\varrho(\psi, H) = det(\psi I - W(H)), \qquad (62)$$

where $W(H)$ is given in (61), the characteristic equation is written as follows

$$\varrho(\psi, H) = q_0(H)\psi^6 + q_1(H)\psi^5 + q_2(H)\psi^4 + q_3(H)\psi^3 + q_4(H)\psi^2 + q_5(H)\psi + q_6(H). \qquad (63)$$

NUMERICAL RESULTS

To illustrate the numerical efficiency of the new IRKD5 method, a set of problems is tested numerically. The new IRKD5 method is compared with the well-known RK methods in the scientific literature. In the numerical comparisons, we used the criterion which is the test based on computing the maximum global error (Log_{10}(Max Error)) versus the computational effort measured by Log_{10}(Function Evaluations) required by each problem as demonstrated in Figures 1- 4. The methods chosen in the numerical experiments are as follows:

- **IRKD5:** The fifth-order IRKD method with four-stage constructed in this paper.

- **RK5:** The fifth-order Runge-Kutta method given in Dormand et al. [14].

- **RKD5:** The fifth-order RKD method with three-stage defined by Mechee et al. [6].

Problem 1: The inhomogeneous linear problem given in [15].

$$y''' = y + \cos(x), \quad 0 < x \le 1$$

with initial condition,

$$y(0) = 0, \ y'(0) = 0, \ y''(0) = 1.$$

The exact solution:

$$y(x) = \frac{1}{2}(e^x - \cos(x) - \sin(x)).$$

Problem 2:

$$y''' = -e^{-x}, \quad 0 < x \leq 1$$

with initial condition,

$$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1.$$

The exact solution: $y(x) = e^{-x}$.

Problem 3:

$$y''' = (12x - 8x^3)y, \quad 0 < x \leq 1$$

with initial condition,

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2.$$

The exact solution: : $y(x) = e^{-x^2}$.

Problem 4: The nonlinear system

$$y''' = z + \frac{1}{\sqrt{u^2+z^2}} - \frac{1}{\sqrt{y+z^2}},$$

$$z''' = -y + \frac{1}{\sqrt{u^2+z^2}} - \frac{1}{\sqrt{y+z^2}},$$

$$u''' = z + \frac{1}{\sqrt{u^2+z^2}} - \frac{1}{\sqrt{y+z^2}}, \quad 0 < x \leq 1$$

with initial condition,

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1,$$

$$z(0) = 0, \quad z'(0) = 1, \quad z''(0) = 0,$$

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1.$$

The exact solution:

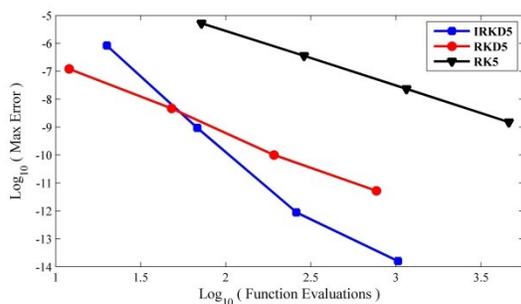$$y(x) = \cos(x), \quad z(x) = \sin(x), \quad u(x) = \cos(x).$$

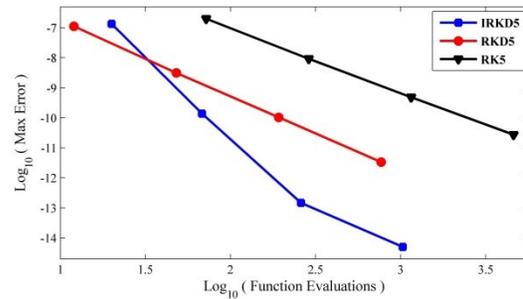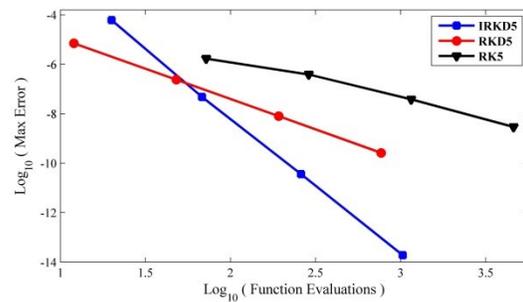Figure 2. The efficiency graphs for Problem 2 with $h = 1/2^i, \quad i = 2,4,6,8.$

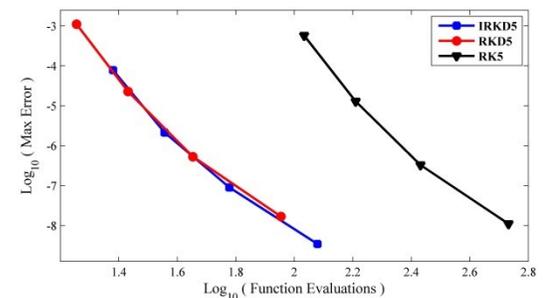Figure 3. The efficiency graphs for Problem 3 with $h = 1/2^i, \quad i = 2,4,6,8.$

Figure 4. The efficiency graphs for Problem 4 with $h = 0.8/2^i, \quad i = 0,1,2,3.$

Figure 1. The efficiency graphs for Problem 1 with $h = 1/2^i, \quad i = 2,4,6,8.$

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CONCLUSION

An improved Runge-Kutta IRKD method for direct solution of special third-order ODEs is constructed in this paper. The order conditions for the IRKD method were derived and based on the order conditions, we have obtained the fifth-order IRKD method with four-stage, denoted as IRKD5. The stability polynomial of IRKD method is presented. Numerical experiments are performed on several problems and the numerical results show the robustness and competency of the new IRKD5 method when it is compared with the RK5 method and direct RKD5 method in terms of error accuracy and number of function evaluations. For all problems IRKD5 method gives the smallest error for the same number of function evaluations followed by RKD5 and RK5 methods. Hence, it can be concluded that IRKD5 method is the most efficient method compared to RKD5 method and RK5 method in solving special third-order ODEs.

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