

## APPROXIMATE SOLUTION OF VACCINATION, SUSCEPTIBLE, EXPOSED, INFECTED, RECOVERED (VSEIR) MODEL OF TUBERCULOSIS IN NORTH SUMATERA INDONESIA

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**ABSTRACT** In this paper, a vaccination susceptible exposed infected recovered (VSEIR) model for tuberculosis in North Sumatera is discussed. The VSEIR model is formed by a system of nonlinear differential equations. The approximate solution of this model is obtained by using the step variational iteration method (SVIM) and variational iteration method (VIM). VIM used the general Lagrange multiplier in the correction functional and runs iteratively. SVIM also used the general Lagrange multipliers for construction of the correction functional for the problems, and runs by step approach, where the computation is done by dividing the interval into subintervals with time steps. The two methods are alternative methods to obtain the approximate solutions of the VSEIR model. Comparison is made with the conventional numerical method, the fourth-order Runge–Kutta method (RK4). From the results, the SVIM solution is more accurate than the VIM solution for long time interval when compared to RK4.

**(Keywords:** VSEIR model, General Lagrange Multiplier, Variational Iteration Method, Step Variational Iteration Method, Fourth Order Runge Kutta)

### INTRODUCTION

Tuberculosis (TB) is an airborne infection and most commonly affects the lungs. TB is a bacterial disease caused by Mycobacterium Tuberculosis, which is transmitted through contaminated air released during coughing of TB patients. TB disease can affect anyone and anywhere, and generally in children the source of infection is from adult TB patients [1]. TB infection can infect virtually the whole body because the bacteria can spread through the blood vessels or lymph nodes. Although the organs most commonly affected are the lungs, in people with low immune systems, it can infect the brain, kidneys, gastrointestinal tract, bone, lymph nodes, etc. [1].

TB is a public health problem in the world despite the efforts to control and the DOTS strategy has been implemented in many countries since 1995. In a WHO report of 2013, there were an estimated 8.6 million TB cases in 2012 where 1.1 million people (13%) of them are in the African region. There are 450,000 people suffering TBMDR and 170,000 of them dead [2]. In North Sumatera, in 2012, around 82.67 % of BTA + (infected) for 17,459 patients and around 83.34 % from the total patients of TB can be cured. To see the development of transmission of Tuberculosis and the dynamics, Rangkuti et al. [5] have built a new model. The VSEIR model is divided into five classes. The class  $V$  represents vaccination,  $S$  represents the susceptible that do not have the disease,  $E$  represents

the exposed that are infected but is yet to show any sign of symptoms,  $I$  represents the infective that have the disease and can transmit it to others,  $R$ , denotes the recovered class of those who went through infection and emerge with permanent or temporary infection-acquired immunity. The VSEIR model is described by the following dynamic system:

$$\dot{V} = qN - \delta_1 V S - \mu_1 V, \quad (1)$$

$$\dot{S} = \delta_1 V S - \mu_2 S - \alpha ES, \quad (2)$$

$$\dot{E} = \alpha ES - \mu_3 E - \rho EI, \quad (3)$$

$$\dot{I} = \rho EI - \beta I, \quad (4)$$

$$\dot{R} = \delta_4 I - \gamma R, \quad (5)$$

where  $\beta = \mu_4 + \mu_{TB} + \delta_4$ . Here, the human birth in natural through passive vaccination ( $V(t)$ ) at rate  $p$ , nonnegative parameters  $\mu_1, \mu_2, \mu_3, \mu_4$  and  $\gamma$  denote as natural death of population of the  $V$ , the  $S$ , the  $E$ , the  $I$  and the  $R$ , respectively. Population of infected Tuberculosis died in rate  $\mu_{TB}$ . The susceptible population decreased due to coming individual from the  $V$  in rate  $\delta_1$ .  $\alpha$  denotes the transfer rate from susceptible to infected population. Infected population increases due to movement of individuals from infected individuals  $I$  in rate  $\delta_4$  and decreased due to movement of individuals in to the  $R$  at rate  $\gamma$ . The model can be simplified by assuming the fractions  $u = \frac{V}{N}, w = \frac{S}{N}, x = \frac{E}{N}, y = \frac{I}{N}$ , and  $z = \frac{R}{N}$

Thus, the model for TB can be simplified as follows

$$\frac{du}{dt} = q - \mu_1 u - \delta_1 u w, \tag{6}$$

$$\frac{dw}{dt} = (\delta_1 u - \mu_2 - \alpha x) w, \tag{7}$$

$$\frac{dx}{dt} = (\alpha w - \mu_3 - \rho y) x, \tag{8}$$

$$\frac{dy}{dt} = \rho x y - \beta y, \tag{9}$$

$$\frac{dz}{dt} = \delta_4 y - \gamma z, \tag{10}$$

subject to initial conditions

$$u(0) = 0.02167; w(0) = 0.8174; x(0) = 0.1583; y(0) = 0.0017 \text{ and } z(0) = 0.00092.$$

The above initial conditions were obtained from the real data for TB in North Sumatera [8].

Most model of real life problems, however, are still very difficult to solve. Our motivation for this work is to provide an alternative analytical method to find the solution for the epidemic model. This is centralized on a newly modified version of VIM, which is generally called the step variational iteration method (SVIM) proposed by Yulita Molliq et al. [3]. We will present comparative solutions from VIM and fourth-order Runge–Kutta method (RK4). We choose the conventional RK4 as our benchmark, as it is widely accepted and exactly used. The accuracy of SVIM has been shown for solving two chaotic systems i.e. Rössler and Genesio systems. In SVIM technique, each interval on VIM is divided into subintervals with time span  $t$  and the solution at each subinterval will be obtained. It is necessary to satisfy the initial conditions at each subinterval, where the initial conditions will be changed for each subinterval. Yulita Molliq et al. [4] modified the SVIM to find the approximate solution of a fractional biochemical reaction model.

This paper is organized as follows: Section 2 discusses the main idea on VIM and some recent advances on the technique, Section 3 deals with the concept of SVIM, Section 4 outlines the application of SVIM to the VSEIR model, Section 5 present the results obtained by the methods mentioning with some critical discussions, and lastly, Section 6 offers some concluding remarks on the method used.

### VARIATIONAL ITERATION METHOD (VIM)

To introduce the basic concepts of variational iteration method (VIM), we consider the following nonlinear differential equation:

$$Lu_i(t) + Nu_i(t) = g_i(t), \tag{11}$$

where  $L$  is a linear operator,  $N$  is a nonlinear operator, and  $g_i(t)$  is an inhomogeneous term  $i = 1, 2, \dots, n$ .

According to VIM, one can construct a correction functional as follows:

$$u_{i,k+1} = u_{i,k} + \int_0^t \lambda_i (Lu_i + Nu_i - g_i) d\xi \tag{12}$$

where  $\lambda_i, i = 1, 2, 3, \dots, n$  are the Lagrange multipliers [6], which can be identified optimally via the variational theory,  $u_{i,n}(\xi)$  and are considered as restricted variations, i.e.  $\delta u_{i,n}(\xi) = 0$ . Once we have determined the Lagrange multiplier, we use VIM to perform the iteration using the initial approximation, which we choose by a linearized solution of the equation that satisfies the initial conditions. Therefore, we can successively approximate or even reach the exact solution by using

$$u(t) = \lim_{n \rightarrow \infty} u_{i,n}(t) \tag{13}$$

where  $n$  is the iteration step.

### STEP VARIATIONAL ITERATION METHOD (SVIM)

In this section, we shall now look at how this new modification of VIM so called step variational iteration method (SVIM) to find the approximate solution for longer time span  $t$ . Here, interval  $[0, T]$  is regarded as an interval, then the interval is divided to subinterval with time span  $t$  and the solution at each subinterval of Eqs. (7)-(10) will be obtained. It is necessary to satisfy the initial condition at each subinterval, the initial conditions  $u_{1,0}, u_{2,0}, \dots, u_{m,0}$  will be changed for each subinterval, i.e. and it should be satisfied through initial conditions,  $u_{i,n}(t^*) = 0$  for all  $n \geq 1$ . Thus the formula can be written as [3]:

$$u_{i,k+1} = u_{i,k} + \int_0^{t-t^*} \lambda_i (Lu_i + Nu_i - g_i) d\xi \tag{14}$$

Here,  $t - t^*$  as upper limit of integration instead of fixed upper limit of  $t$  in Eq. (12). The approximate solution takes the form:

$$u_i(t) \approx u_{i,n}(t - t^*) \tag{15}$$

Where  $t^*$  start from  $t_0 = 0$  until  $t_j = T, j$  is number of subinterval. To carry out the solution on every subinterval of equal length  $\Delta t$  the values of the following initial condition are shown below

$$c_i^* = u_i(t^*), i = 1, 2, \dots, m. \tag{16}$$

In general, we do not have the information of our clearance except at the initial point  $t = t_0 = 0$ , but these values can be obtained by assuming that the new initial condition is the solution in previous interval i.e. if the solution in interval  $[t_j, t_{j+1}]$  is necessary then the initial condition of this interval will be as follows

$$c_i = u_i \approx u_{i,n}(t_j, t_{j-1}), \tag{17}$$

where  $c_i, i = 1, 2, \dots, m$  are the initial conditions in the interval  $[t_j, t_{j+1}]$ .

**APPLICATION**

In this section, the VIM method is applied to compute an approximate solution of nonlinear system of differential equations describing a VSEIR model [5]. According to VIM the correctional functional constructing as follow

$$u_{k+1} = u_k + \int_0^t \lambda_1 \left( \frac{du_k}{d\xi} - q + \mu_1 u_k + \delta_1 u_k w_k \right) d\xi, \tag{18}$$

$$w_{k+1} = w_k + \int_0^t \lambda_2 \left( \frac{dw_k}{d\xi} - ((\delta_1 u_k - \mu_2 - \alpha x_k) w_k) \right) d\xi, \tag{19}$$

$$x_{k+1} = x_k + \int_0^t \lambda_3 \left( \frac{dx_k}{d\xi} - ((\alpha w_k - \mu_3 - \rho y_k) x_k) \right) d\xi, \tag{20}$$

$$y_{k+1} = y_k + \int_0^t \lambda_4 \left( \frac{dy_k}{d\xi} - (\rho x_k y_k - \beta y_k) \right) d\xi, \tag{21}$$

$$z_{k+1} = z_k + \int_0^t \lambda_5 \left( \frac{dz_k}{d\xi} - (\delta_4 y_k - \gamma z_k) \right) d\xi, \tag{22}$$

where  $\lambda_i, i = 1, 2, 3, \lambda_i$  are the general Lagrange multiplier which can be identified optimally via the variational theory and the subscript  $k$  indicates the  $n$ th. To obtain the optimal  $\lambda_i(\xi)$ , we proceed as follows:

$$\delta u_{k+1} = \delta u_k + \int_0^t \delta \lambda_1 \left( \frac{du_k}{d\xi} - q + \mu_1 u_k + \delta_1 \widehat{u_k w_k} \right) d\xi, \tag{23}$$

$$\delta w_{k+1} = \delta w_k + \int_0^t \delta \lambda_2 \left( \frac{dw_k}{d\xi} - (\delta_1 \widehat{u_k w_k} - \mu_2 w_k - \alpha \widehat{x_k w_k}) \right) d\xi, \tag{24}$$

$$\delta x_{k+1} = \delta x_k + \int_0^t \delta \lambda_3 \left( \frac{dx_k}{d\xi} - (\alpha \widehat{w_k x_k} - \mu_3 x_k - \rho \widehat{y_k x_k}) \right) d\xi, \tag{25}$$

$$\delta y_{k+1} = \delta y_k + \int_0^t \delta \lambda_4 \left( \frac{dy_k}{d\xi} - (\rho \widehat{x_k y_k} - \beta y_k) \right) d\xi, \tag{26}$$

$$\delta z_{k+1} = \delta z_k + \int_0^t \delta \lambda_5 \left( \frac{dz_k}{d\xi} - (\delta_4 \widehat{y_k} - \gamma z_k) \right) d\xi, \tag{27}$$

where  $\widehat{u_k w_k}, \widehat{x_k y_k}, \widehat{x_k w_k}$  and  $\widehat{y_k}$  are considered restricted variations i.e.  $\delta u_k w_k = 0, \delta x_k y_k = 0, \delta x_k \widehat{w_k} = 0$  and  $\delta \widehat{y_k} = 0$ , Then we have

$$\delta u_{k+1} = \delta u_k + \int_0^t \left( \delta \lambda_1 \frac{du_k}{d\xi} + \delta \lambda_1 \mu_1 u_k \right) d\xi, \tag{28}$$

$$\delta w_{k+1} = \delta w_k + \int_0^t \left( \delta \lambda_2 \frac{dw_k}{d\xi} + \delta \lambda_2 \mu_2 w_k \right) d\xi, \tag{29}$$

$$\delta x_{k+1} = \delta x_k + \int_0^t \left( \delta \lambda_3 \frac{dx_k}{d\xi} + \delta \lambda_3 \mu_3 x_k \right) d\xi, \tag{30}$$

$$\delta y_{k+1} = \delta y_k + \int_0^t \left( \delta \lambda_4 \frac{dy_k}{d\xi} + \delta \lambda_4 \beta y_k \right) d\xi, \tag{31}$$

$$\delta z_{k+1} = \delta z_k + \int_0^t \left( \delta \lambda_5 \frac{dz_k}{d\xi} + \delta \lambda_5 \gamma z_k \right) d\xi, \tag{32}$$

Thus, the general Lagrange multiplier are obtained as follow

$$\lambda_1 = -e^{-\mu_1(\xi-t)} \tag{33}$$

$$\lambda_2 = -e^{-\mu_2(\xi-t)} \tag{34}$$

$$\lambda_3 = -e^{-\mu_3(\xi-t)} \tag{35}$$

$$\lambda_4 = -e^{-\beta(\xi-t)} \tag{36}$$

$$\lambda_5 = -e^{-\gamma(\xi-t)} \tag{37}$$

Here, the general Lagrange multiplier in (34)-(37) is expanded by Taylor series only one term, so the general Lagrange multiplier can be written as follows

$$\lambda_1 = -1 \tag{38}$$

$$\lambda_2 = -1 \tag{39}$$

$$\lambda_3 = -1 \tag{40}$$

$$\lambda_4 = -1 \tag{41}$$

$$\lambda_5 = -1 \tag{42}$$

Substitute the general Lagrange multiplier into (38)-(42) into the correctional iteration functional in Eqs. (19)-(22) result in the following iteration formula:

$$u_{k+1} = u_k - \int_0^t \left( \frac{du_k}{d\xi} - q + \mu_1 u_k + \delta_1 u_k w_k \right) d\xi, \tag{43}$$

$$w_{k+1} = w_k - \int_0^t \left( \frac{dw_k}{d\xi} - ((\delta_1 u_k - \mu_2 - \alpha x_k) w_k) \right) d\xi, \tag{44}$$

$$x_{k+1} = x_k - \int_0^t \left( \frac{dx_k}{d\xi} - ((\alpha w_k - \mu_3 - \rho y_k)x_k) \right) d\xi, \quad (45)$$

$$y_{k+1} = y_k - \int_0^t \left( \frac{dy_k}{d\xi} - (\rho x_k y_k - \beta y_k) \right) d\xi, \quad (46)$$

$$z_{k+1} = z_k - \int_0^t \left( \frac{dz_k}{d\xi} - (\delta_4 y_k - \gamma z_k) \right) d\xi, \quad (47)$$

The interval  $[0, T]$  is divided to subintervals with the time step  $\Delta t$  to obtain the solution at each subinterval. In this case, the initial conditions is satisfied at each of the subinterval [7], i.e  $u(t^*) = c_1^* = u_0, w(t^*) = c_2^* = w_0, x(t^*) = c_3^* = x_0, y(t^*) = c_4^* = y_0$  and  $z(t^*) = c_5^* = z_0$ . The initial conditions should be satisfied  $u_k(t^*) = 0, w_k(t^*) = 0, x_k(t^*) = 0, y_k(t^*) = 0$  and  $z_k(t^*) = 0$  for all  $n \geq 1$ . Such that (7) to (10) can be written as

$$u_{k+1} = u_k - \int_0^{t-t^*} \left( \frac{du_k}{d\xi} - q + \mu_1 u_k + \delta_1 u_k w_k \right) d\xi, \quad (48)$$

$$w_{k+1} = w_k - \int_0^{t-t^*} \left( \frac{dw_k}{d\xi} - ((\delta_1 u_k - \mu_2 - \alpha x_k)w_k) \right) d\xi, \quad (49)$$

$$x_{k+1} = x_k - \int_0^{t-t^*} \left( \frac{dx_k}{d\xi} - ((\alpha w_k - \mu_3 - \rho y_k)x_k) \right) d\xi, \quad (50)$$

$$y_{k+1} = y_k - \int_0^{t-t^*} \left( \frac{dy_k}{d\xi} - (\rho x_k y_k - \beta y_k) \right) d\xi, \quad (51)$$

$$z_{k+1} = z_k - \int_0^{t-t^*} \left( \frac{dz_k}{d\xi} - (\delta_4 y_k - \gamma z_k) \right) d\xi, \quad (52)$$

### RESULTS AND DISCUSSION

The Maple mathematical software was used for all our computations. The iterative schemes for VIM in (44)–(47) and SVIM in (49)–(52) are coded in the computer algebra package Maple and we employed Maple’s built in fourth-order Runge–Kutta procedure rk4. We revised the parameters and initial conditions in [5] due to the updating of data for TB in North Sumatera in which the parameters are determined from previous studies and the Health Department of North Sumatera province as shown in Table 1.

**Table 1.** Parameter Values

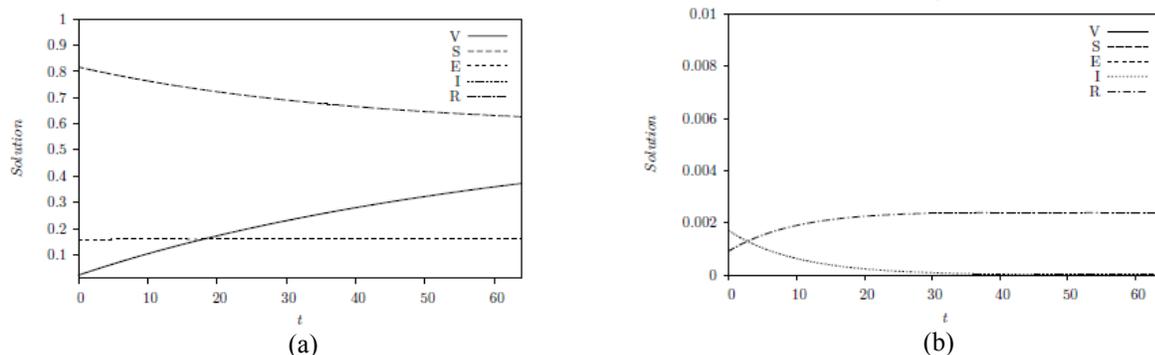
Name of Parameter	Values	Reference
birth rate ( $q$ )	0.0094	[8]
death rate for baby ( $\mu_1$ )	0.0065	[8]
rate of baby vaccine convert to susceptible ( $\delta_1$ )	0.0160	[8]
death rate ( $\mu_2$ )	0.0075	[8]
rate of susceptible to exposed ( $\alpha$ )	0.0016	[8]
death rate ( $\mu_3$ )	0.0009	[9]
rate of exposed to infected ( $\rho$ )	0.00859	[8]
death rate cause TB ( $\mu_{TB}$ )	0.0071	[8]
death rate ( $\mu_4$ )	0.0009	[9]
rate of infected to recovery ( $\delta_4$ )	0.0919	[8]
death rate in recovery period ( $\gamma$ )	0.0009	[9]

Here, the number of population ( $N$ ) is 13,215,401 including all babies born i.e. 303,327 babies, number of vaccination is 284,633. The number of susceptible is 10,802,233, exposed is 2114464, infected is 22,360, recovery is 12154, and mortality due to TB is 117. Thus the initial conditions used are  $u(0) = 0.02167, w(0) = 0.8174, x(0) = 0.1583,$

$y(0) = 0.0017$  and  $z(0) = 0.00092$  for all computations. We determine the accuracy of RK4 for the solution of Model in (7)-(10) shown in Figure 1, since the RK4 is widely used. This solution is viewed as a benchmark for this model. We used 4 iterate VIM and SVIM to find the spreading number of vaccination ( $u$ ), susceptible ( $w$ ), exposed ( $x$ ),

infected ( $y$ ), and recovery ( $z$ ) probability at time step  $\Delta t = 0.0001$ , respectively. The comparisons displayed between results from VIM, SVIM and RK4 for  $t \in [0, 60]$ , in figure 2. From the figure, it is obvious that VIM exhibits unpredictable behaviour because the graph diverts from the RK4 and SVIM. Both RK4 and SVIM solutions show good

synchronization at the time carried out and both the results agree very well with each other. We note that the solution of all variables will converge to RK4 solution in certain time. Table 2 presents the absolute value of 4<sup>th</sup> iterate SVIM and VIM for recovery case( $z(t)$ ).



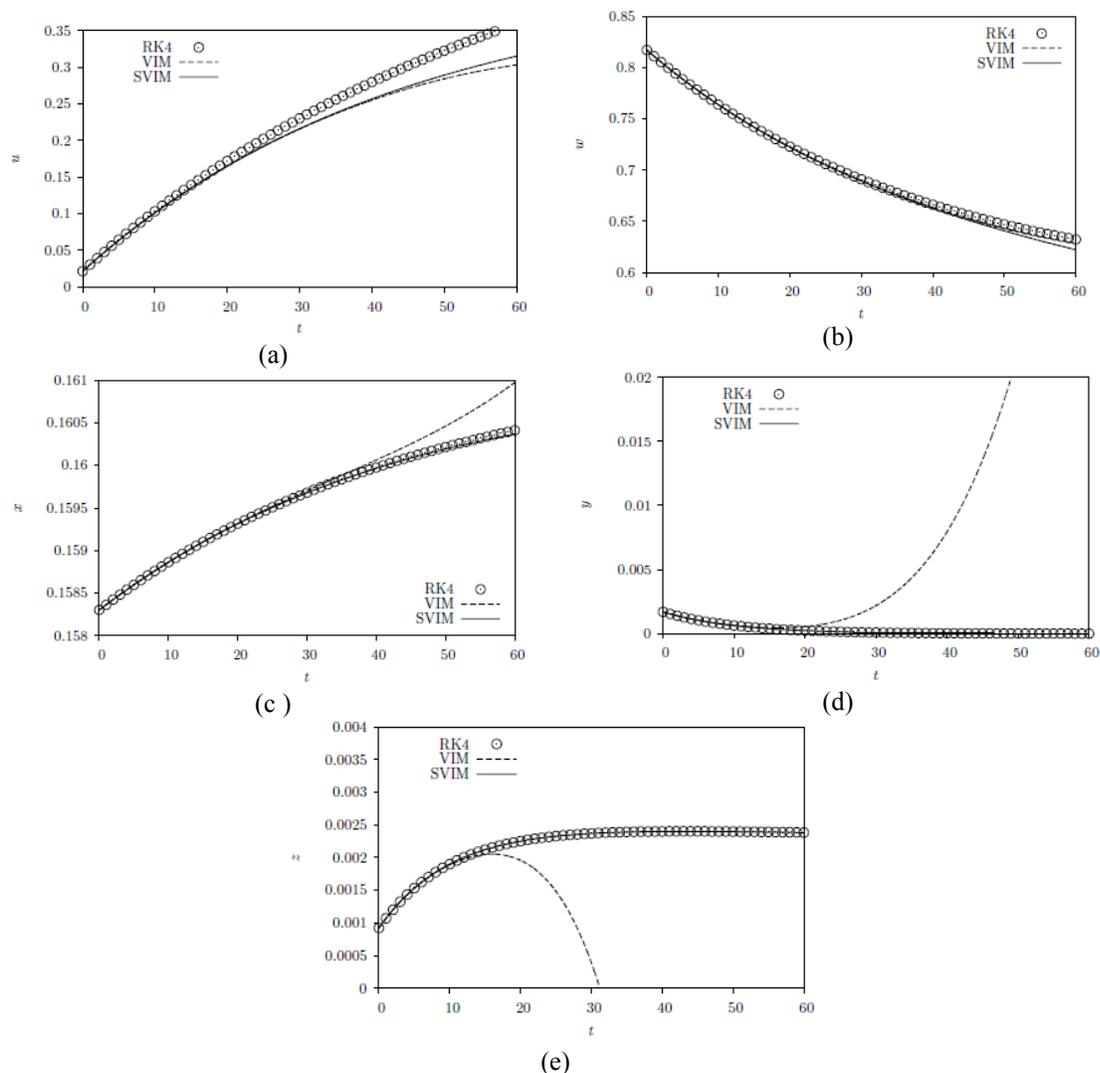
**Figure 1.** The number of spreading of Tuberculosis; (a)  $u(t), w(t), x(t)$ , (b)  $y(t), z(t)$  which are obtained using RK4 for  $\Delta t = 0.0001$ .

**Table 2.** The Absolute Error of 4<sup>th</sup> Iterate SVIM Solutions compared to RK4 with  $\Delta t = 0.001$

$t$	SVIM				
	$\Delta u$	$\Delta w$	$\Delta x$	$\Delta y$	$\Delta z$
5	6.619 E-04	1.190 E-05	9.456 E-08	5.908 E-11	6.061 E-07
10	2.032 E-03	8.562 E-05	2.326 E-07	6.043 E-11	9.736 E-07
15	4.102 E-03	2.578 E-04	5.142 E-07	2.504 E-11	1.195 E-06
20	6.833 E-03	5.607 E-04	1.046 E-06	1.677 E-11	1.328 E-06
25	1.017 E-02	1.021 E-03	2.062 E-06	1.235 E-11	1.407 E-06
30	1.403 E-02	1.662 E-03	3.779 E-06	4.940 E-12	1.452 E-06
35	1.836 E-02	2.502 E-03	6.441 E-06	3.878 E-12	1.477 E-06
40	2.307 E-02	3.554 E-03	1.032 E-05	1.154 E-11	1.490 E-06
45	2.809 E-02	4.83 E-03	1.569 E-05	1.627 E-11	1.495 E-06
50	3.335 E-02	6.338 E-03	2.284 E-05	1.861 E-11	1.495 E-06

Table 2 shows the accuracy of VIM and the maximum error of VIM solution is  $|10^{-02}|$ . The solutions of SVIM (4-iterates) are compared to those of RK4. The maximum error using SVIM is now decreased to  $|10^{-06}|$  if it is compared to the maximum error using VIM. It also occurs in the infected case ( $y(t)$ ),

where the maximum error of VIM solution is  $|10^{-02}|$ , whereas the maximum error of SVIM solution has decreased i.e.  $|10^{-11}|$ ; see Table 2. This shows that the SVIM has better accuracy.



**Figure 2.** The approximate solution of; (a)  $u(t)$ , (b)  $w(t)$ , (c)  $x(t)$ , (d)  $y(t)$ , (e)  $z(t)$  which are obtained using SVIM, VIM and RK4

**Table 3.** The Absolute Error of 7<sup>th</sup> Iterate VIM Solutions compared to RK4 with  $\Delta t = 0.001$

$t$	VIM				
	$\Delta u$	$\Delta w$	$\Delta x$	$\Delta y$	$\Delta z$
5	6.619 E-04	1.717 E-05	1.865 E-09	3.798 E-07	3.574 E-07
10	2.033 E-03	9.396 E-05	6.975 E-08	1.127 E-05	1.061 E-05
15	4.115 E-03	2.613 E-04	6.951 E-07	7.970 E-05	7.502 E-05
20	6.889 E-03	5.376 E-04	3.066 E-06	3.139 E-04	2.955 E-04
25	1.034 E-02	9.267 E-04	9.280 E-06	8.985 E-04	8.458 E-04
30	1.445 E-02	1.416 E-03	2.245 E-05	2.104 E-03	1.980 E-03
35	1.924 E-02	1.977 E-03	4.679 E-05	4.291 E-03	4.039 E-03
40	2.476 E-02	2.564 E-03	8.764 E-05	7.916 E-03	7.452 E-03
45	3.109 E-02	3.115 E-03	1.515 E-04	1.353 E-02	1.274 E-02
50	3.834 E-02	3.552 E-03	2.459 E-04	2.179 E-02	2.051 E-02

## CONCLUSION

In this paper, an algorithm of VSEIR model for TB using the step variational iteration method (SVIM) was implemented. We found that SVIM is a suitable technique to solve the system of nonlinear differential equations. This method yields a solution by iterations of a rapid convergent infinite power series with enlarged intervals. Comparison between SVIM, VIM and RK4 were made; the SVIM was found to be more accurate than the VIM. SVIM is easier for calculation yet powerful and is also readily applicable to the more complex cases of problems which arise in various fields of pure and applied sciences.

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