Malaysian Journal of Science **17B**: 69-74 (1996)

On coastal circulation modeling

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ABSTRACT This paper reviews the implementation of a generalized vertical coordinate system leading to a σ vertical coordinate system for the ocean. The theoretical framework, of a new technique that splits the barotropic and baroclinic modes is discussed.

ABSTRAK Kertas ini mengulas penggunaan sistem koordinat menegak umum yang menjurus kepada sistem koordinat menegak σ untuk lautan. Satu teknik baru yang mengasingkan mod barotropik dan baroklinik dibincangkan secara teoritikal.

INTRODUCTION

The external mode of the wind-driven ocean circulation of the South China Sea has been modeled; as an extension to this study [1], the theoretical framework of a new method that combines both the internal and the external modes is described.

The first part of this report describes the equations that govern the ocean general circulation in the (x,y,z) coordinate system. This coordinate system is then transformed to the generalized coordinate system (x,y,s), where s is the new generalized coordinate [2]. Because of the fact that the same number of layers is needed in coastal areas as well as in deep ones, a stretching vertical coordinate is implemented. This study, introduces a σ vertical coordinate so that the set of equations can be transformed into the (x,y,σ) coordinate system [3,4].

The mode splitting technique is also described [4,5]. This method is effective in the interaction of both the external and internal modes.

MODEL EQUATIONS

Wind-driven ocean circulation is fairly well described by both the hydrodynamic laws of conservation of mass and momentum. The thermodynamic and the salinity equations, as well as the equation of state are needed to describe the thermohaline circulation. This set of six equations form the basic equations that govern most ocean circulation models.

A. Basic equations

The model equations, in the x, y and z coordinate system are:

1) The horizontal momentum equation

$$dV/dt + fKxV = -\rho^{-1}\nabla p + F + \nabla.(A_h\nabla V) + \partial[A_v\partial V/\partial z]/\partial z \quad (1)$$

where

$V = u\mathbf{I} + v\,\mathbf{J},$
$\nabla(\ ) = \partial(\ )/\partial x\mathrm{I} + \partial(\ )/\partial y\mathrm{J},$
$d(\ )/dt = \partial(\ )/\partial t + V.\nabla(\ ) + w\partial(\ )/\partial z,$

I, **J** and **K** (u, v, and w), represent the unit vectors (velocity components) in the x, y, and z directions, respectively; ***F***, the body force; f, the Coriolis parameter; t, the time; ρ, the density; p, the pressure; A_h (A_v), the horizontal (vertical) kinematic turbulent eddy viscosity coefficient, and the symbol x (.) the vectorial (scalar) product.

2) The vertical equation of motion

Because of the large-scale flow approximation that has been implemented, the vertical equation of motion may be expressed by the hydrostatic approximation:

$$\partial p/\partial z = -\rho g \quad (2)$$

where g, represents the earth' s gravity.

3) The continuity equation

$$\partial \rho/\partial t + \nabla.(\rho V) + \partial(\rho w)/\partial z = 0 \quad (3)$$

4) The heat temperature equation

This thermodynamic equation, derived from the first law of thermodynamics, yields

$$\partial T/\partial t = k\nabla^2 T + Q/C_p \tag{4}$$

where T, represents the temperature; k, the kinematic thermal diffusivity coefficient; ∇^2 T, the Laplacian of temperature; C_p, the coefficient of specific heat at constant pressure; Q, the adiabatic heating.

5) The salinity equation

$$dS/dt = \gamma(S), \tag{5}$$

where S represents salinity and $\gamma(S)$ is a function of salinity.

6) The equation of state for the ocean

$$\rho = \rho\,(T,S) \tag{6}$$

B. Boundary Conditions

1) At the free surface

The acting body forces representing the top boundary condition at the free surface are

$$\rho D(\partial V/\partial z) = \Gamma^w \tag{7}$$

where Γ^w represents the wind stress vector. On the other hand, the kinematic condition states that

$$w = \partial\eta/\partial t + u\partial\eta/\partial x + v\partial\eta/\partial y \tag{8}$$

where η represents the free surface elevation.

2) At the lower boundary

The bottom friction dissipation is parameterized by the relation

$$\rho D(\partial V/\partial z) = \Gamma_b \tag{9}$$

where Γ_b, represents the bottom friction stress; D [= H(x,y) + η (x,y,t)], the total depth of the water column; and H(x,y), the mean depth of the water column.

The vertical velocity at the bottom is parameterized by

$$w = -(u_b\partial H/\partial x + v_b\partial H/\partial y) \tag{10}$$

where (u_b, v_b) is the bottom friction velocity associated with bottom frictional stress Γ_b.

If the terms ***F***, Q, and $\gamma(S)$ are functions of the dependent variables ***V***, w, p and ρ, the system of Eqs. (1) - (6), in addition to the boundary conditions, constitutes what is known as a *closed system of equations*. The system of Eqs. (1) - (6) are the model equations most commonly used in ocean circulation models.

In most cases, the evolution of the water mass flux is accurately predicted by the time integration of the model equations above stated. In most problems in physical oceanography, models are set to start from rest; i.e., u = v = η = 0. Owing to the fact that the equations are non-linear, an analytical solution is difficult to obtain.

GENERALIZED COORDINATE SYSTEM (S)

The oceanic vertical structure may be represented using different vertical coordinate systems. The vertical coordinate systems may be classified as either Eulerian (z or σ) or Langrangian (ρ). In the former case, the velocity (speed and direction) of the fluid particle at every point in the fluid at every instant of time, is described. In the latter case, the trajectories followed by each fluid particle, when each particle reaches each point in its path, is described.

The transformation of the vertical coordinate z to the generalized vertical coordinate s is given by

$$s = s(x,y,z,t) \tag{11}$$

Any scalar function, A, may be a function of in terms of z or s, depending on the particular choice of the vertical coordinate system been implemented. The partial derivative of A with respect to c (where c may represent either x, y, or t), yields

$$(\partial A/\partial c)_s = (\partial A/\partial c)_z + (\partial A/\partial z)(\partial z/\partial c)_s \tag{12}$$

where the subscript represents the vertical coordinate system that remains constant during the partial differentiation. In using the relation

$$\partial A/\partial z = (\partial s/\partial z)(\partial A/\partial s) \tag{13}$$

Eq. (12) yields

$$(\partial A/\partial c)_s = (\partial A/\partial c)_z + (\partial A/\partial s)(\partial s/\partial z)(\partial z/\partial c)_s \quad (14)$$

For c = t, it follows that

$$(\partial A/\partial t)_s = (\partial A/\partial t)_z + (\partial A/\partial s)(\partial s/\partial z)(\partial z/\partial t)_s \quad (15)$$

If, on the other hand, c = x,y

$$\nabla_s A = \nabla_z A + \nabla_s z(\partial A/\partial s)(\partial s/\partial z) \quad (16)$$

Taking into consideration the system of Eqs. (11) - (16), the total derivative, d( )/d t, in the generalized vertical coordinate system, s, is

$$d(\ )/dt = \partial(\ )/\partial t + V.\nabla_s(\ ) + [w - (\partial z/\partial t)_s - \nabla_s z](\partial s/\partial t)\{\partial(\ )/\partial s\} \quad (17)$$

The total derivative in the (x,y,s) system is

$$d(\ )/dt = \{\partial(\ )/\partial t\}_s + V.\nabla_s(\ ) + \dot{s}(\partial(\ )/\partial s), \quad (18)$$

where

$$\dot{s} = dx/dt \quad (19)$$

Upon comparison of Eqs. (17) - (18), it follows that

$$\dot{s} = d(\ )/dt = (\partial s/\partial z)[w - (\partial z/\partial t)_s - V.\nabla_s z] \quad (20)$$

It also follows that the horizontal momentum equation in the (x,y,s) system, without introducing the eddy viscosity terms, is

$$dV/dt + fKx\,V = -\rho^{-1}\nabla_s p + \rho^{-1}(\partial s/\partial z)\nabla_s z(\partial p/\partial s) + F \quad (21)$$

Therefore, in making use of the hydrostatic relation, it follows that

$$dV/dt + fKx\,V = -\rho^{-1}\nabla_s p - g\nabla_s z + F \quad (22)$$

The continuity equation in the (x,y,s) system, yields

$$\partial w/\partial z = (\partial s/\partial z)[\{d(\partial z/\partial s)/dt\} + \{(\partial V/\partial s).\nabla_s z\}] + [\partial \dot{s}/\partial s] \quad (23)$$

The velocity divergence, following (2.6) yields

$$\nabla_z.V = \nabla_s.V - \nabla_s.z(\partial V/\partial s)(\partial s/\partial z) \quad (24)$$

Upon substitution of Eqs. (23) and (24) in (3), the continuity equation in the (x,y,s) system is

$$[\partial(\rho\partial z/\partial s)/\partial z]_s + \nabla_s.\{\rho V(\partial z/\partial s)\} + [\partial\{(\rho\dot{s})(\partial z/\partial s)\}/\partial s] = 0 \quad (25)$$

In considering the hydrostatic equation in the (x,y,s) system, it follows that

$$\rho\partial z/\partial s = -g^{-1}\partial p/\partial s \quad (26)$$

The continuity equation may, then, be rewritten as

$$[\partial(\partial p/\partial t)/\partial s] + \nabla_s.\{V(\partial p/\partial s)\} + [\partial\{\dot{s}(\partial p/\partial s)\}/\partial s] = 0 \quad (27)$$

The transformation from the (x,y,z) into new (x,y,s) coordinate system of both the salinity and the temperature equations is given by Eq. (18) in terms of the total derivative in the new coordinate system. The derivation of these two equations is left to the reader.

VERTICAL COORDINATE SYSTEM (σ).

A. Basic Equations

In order to have the same number of layers in coastal areas as in deep waters, the vertical structure of the ocean is usually represented using a stretching vertical coordinate system. The stretching coordinate system most commonly in oceanic modeling is σ. This coordinate system is usually defined as

$$\sigma = \{ z - \eta(x,y,t) \} / [H(x,y) + \eta(x,y,t)]$$

$$= \{ z - \eta(x,y,t) \} / D(x,y,t) \quad (28)$$

where H(x,y) represents the mean depth of the water column, η (x,y,t), the free surface elevation and D (x,y,t) = [H(x,y) + η (x,y,t)], the total depth of the water column.

Given the above definition of σ, it is inferred that the water column between $z = \eta$ and $z = -H$, ranges between $\sigma = 0$ and $\sigma = -1$ in the new vertical system.

From the Eqs. (13) and (14), the kinematic vertical eddy (turbulent) viscosity term, $\partial[A_v \partial V/\partial z]/\partial z$, in the (x,y,σ) system, may be expressed as

$$(\partial\sigma/\partial z)\left\{\partial\left[A_v(\partial\sigma/\partial z)(\partial u/\partial\sigma)\right]/\partial\sigma\right\} = D^{-2}\partial\left\{A_v(\partial u/\partial\sigma)/\partial\sigma\right\} \tag{29}$$

From Eq. (16), the pressure gradient, yields

$$\nabla_z p = \nabla_\sigma p - D^{-1}[\nabla_\sigma\eta + \sigma\nabla_\sigma D](\partial p/\partial\sigma) \tag{30}$$

Following (26), the hydrostatic equation in the (x,y,σ) system is

$$(\partial p/\partial\sigma)(\partial\sigma/\partial z) = -\rho g \tag{31}$$

The vertical integration of (31) yields

$$\int_\sigma^0[\partial p/\partial\sigma]\partial\sigma = -\int_\sigma^0 \rho g D d\sigma \tag{32}$$

Therefore

$$p(\sigma) = \int_\sigma^0 \rho g D d\sigma \tag{33}$$

Application of the gradient ∇_z to ρ D, ensues that

$$\nabla_z(\rho D) = \nabla_\sigma(D\rho) + D^{-1}[\nabla_\sigma h + \sigma\nabla_\sigma D][\partial(\rho D)/\partial\sigma] \tag{34}$$

The operator ∇_z [p (σ)] has the form

$$\begin{aligned}\nabla_z[p(\sigma)] &= g\int_\sigma^0 \nabla_z(\rho D) \\ &= g\rho\int_\sigma^0 \nabla_\sigma(D)d\sigma + g\int_\sigma^0 D\nabla_\sigma(\rho)\,d\sigma \\ &+ g\nabla_\sigma(D)\int_\sigma^0 \sigma[\partial(\rho)/\partial\sigma]\,d\sigma + \\ &g\nabla_\sigma(\eta)\int_\sigma^0[\partial(\rho)/\partial\sigma]\,d\sigma\end{aligned} \tag{35}$$

The last term is smaller than both the third and the second terms in the RHS. Thus, it is negligible and it is not been considered.

The multiplication of ρ^{-1} and Eq. (35), yields

$$\rho^{-1}\left\{\nabla_z[p(\sigma)]\right\} = g\nabla_\sigma(\eta) + (gD/\rho)\nabla_\sigma\int_\sigma^0(\rho)\,d\sigma + \left[g\rho^{-1}\nabla_\sigma(D)\right]\int_\sigma^0\left\{\sigma[\partial(\rho)/\partial\sigma]\right\}d\sigma \tag{36}$$

This term represents the pressure gradient force in the (x,y,σ) system. The equations of motion in the (x,y,σ) system yield

$$\begin{aligned}&\partial u/\partial t + u[\partial u/\partial x] + v[\partial u/\partial y] + \dot\sigma[\partial u/\partial\sigma] - fv \\ &= -g[\partial\eta/\partial x] + F_x - (gD/\rho)\left\{\partial/\partial x\left[\int_\sigma^0 \rho\, d\sigma\right]\right\} + \\ &\left[g\rho^{-1}(\partial D/\partial x)\right]\int_\sigma^0\left\{\sigma[\partial(\rho)/\partial\sigma]\right\}d\sigma + \partial[A_h(\partial u/\partial x)]/\partial x + \\ &\partial[A_h(\partial u/\partial x)]/\partial y + D^{-2}\partial\left\{A_v(\partial u/\partial\sigma)/\partial\sigma\right\}\end{aligned} \tag{37a}$$

$$\begin{aligned}&\partial v/\partial t + u[\partial v/\partial x] + v[\partial v/\partial y] + \dot\sigma[\partial v/\partial\sigma] + \\ &fu = -g[\partial\eta/\partial y] + F_y - (gD/\rho)\left\{\partial/\partial y\left[\int_\sigma^0 \rho\, d\sigma\right]\right\} + \\ &\left[g\rho^{-1}(\partial D/\partial y)\right]\int_\sigma^0\left\{\sigma[\partial(\rho)/\partial\sigma]\right\}d\sigma + \\ &\partial[A_h(\partial v/\partial x)]/\partial x + \partial[A_h(\partial v/\partial x)]/\partial y + \\ &D^{-2}\partial\left\{A_v(\partial v/\partial\sigma)/\partial\sigma\right\}\end{aligned} \tag{37b}$$

The continuity equation in the (x,y,σ) system yields

$$\partial\eta/\partial t + \partial(Du)/\partial x + \partial(Dv)/\partial y + \partial\left(D\dot\sigma\right)/\partial\sigma = 0 \tag{38}$$

B. Boundary Conditions

The new vertical velocity, $\dot\sigma$, may be derived as follows:

$$w = \partial z/\partial t + V.\nabla_\sigma z + \dot\sigma\,\partial z/\partial\sigma \tag{39}$$

Since $\partial z/\partial\sigma = D$, it follows that

$$\dot{\sigma}D = w - \left(\partial z/\partial t + V.\nabla_\sigma z\right) \quad (40)$$

i.e.,

$$\dot{\sigma}D = w - \partial(\sigma D + \eta)/\partial t - V.\nabla_\sigma(\sigma D + \eta) \quad (41)$$

From (8) and (10), the boundary conditions are

$$\dot{\sigma}(z=\eta) = \dot{\sigma}(z=-H) = 0 \quad (42)$$

BAROTROPIC - BAROCLINIC TECHNIQUE

In coastal circulation modeling, it is useful to separate the vertical integrated equations (external or barotropic) mode from the vertical structure equations (internal or baroclinic) mode. The former mode contains the external fastest gravity waves whereas the latter contains the slowly moving internal gravity waves.

This new method is known as the mode splitting technique [3,4]. This technique divides the flow into its barotropic and its baroclinic modes. Eqs. (37) may then be rewritten as

$$\partial u/\partial t - fv = -g[\partial\eta/\partial x] + E$$
$$-(gD/\rho)\left\{\partial/\partial x\left[\int_\sigma^0 \rho\, d\sigma\right]\right\} + \left[g\rho^{-1}(\partial D/\partial x)\right]\int_\sigma^0\left\{\sigma[\partial(\rho)/\partial\sigma]\right\} d\sigma \quad (43)$$

$$\partial v/\partial t + fu = -g[\partial\eta/\partial y] + G$$
$$-(gD/\rho)\left\{\partial/\partial y\left[\int_\sigma^0 \rho\, d\sigma\right]\right\} + \left[g\rho^{-1}(\partial D/\partial y)\right]\int_\sigma^0\left\{\sigma[\partial(\rho)/\partial\sigma]\right\} d\sigma \quad (44)$$

where

$$E = u[\partial u/\partial x] + v[\partial u/\partial y] + \dot{\sigma}[\partial u/\partial\sigma] + F_x$$
$$- \partial[A_h(\partial u/\partial x)]/\partial x - \partial[A_h(\partial u/\partial x)]/\partial y - D^{-2}\partial\{A_v(\partial u/\partial\sigma)/\partial\sigma\} \quad (45)$$

$$G = +u[\partial v/\partial x] + v[\partial v/\partial y] + \dot{\sigma}[\partial v/\partial\sigma] + F_y -$$
$$\partial[A_h(\partial v/\partial x)]/\partial y - \partial[A_h(\partial v/\partial y)]/\partial y - D^{-2}\partial\{A_v(\partial v/\partial\sigma)/\partial\sigma\} \quad (46)$$

Eqs. (43) and (44) are vertically averaged, in the following manner:

$$[\ ] = (1/H)\int_{-H}^{\eta}(\ )\,dz = \left(D^{-1}\right)\int_{-1}^{0} D(\ )\,d\sigma \quad (47)$$

Therefore, any variable () may then be interpreted in the following manner:

$$(\ ) = [\ ] + (\ )' \quad (48)$$

As usual, it is assumed that η is at least two orders of magnitude smaller than D. In doing so, we will have

$$\partial[u]/\partial t - f[v] = -g\partial[\eta]/\partial x + [E] \quad (49)$$

$$\partial[v]/\partial t + f[u] = -g\partial[\eta]/\partial y + [F] \quad (50)$$

The continuity equation vertically averaged has the form:

$$\partial[\eta]/\partial t + \partial[uD]/\partial x + \partial[vD]/\partial y = 0 \quad (51)$$

As an example, to illustrate further the methodology been employed, the derivation of Eq. (51), is shown in Appendix I. Upon subtraction of Eqs. (49) and (50) from Eqs. (43) and (44) we will have

$$\partial u'/\partial t - fv' = E - [E] \quad (52)$$

$$\partial v'/\partial t + fu' = F - [F] \quad (53)$$

Eqs. (52) and (53) are, therefore, independent of the terms that govern the external gravity waves. We have effectively separated the internal and the external modes. Eqs. (52), (53) with (38) govern the slow moving baroclinic waves and internal waves. Eqs. (43), (44) and (51) govern the external gravity wave modes.

APPENDIX I

Derivation of Eq. (51)

Following (38) the continuity equation has the form

$$\partial\eta/\partial t + \partial(Du)/\partial x + \partial(Dv)/\partial y + \partial(D\sigma)/\partial\sigma = 0 \quad (A.1)$$

The vertical average of the first term of (A.1) is

$$(1/D)\int_{-1}^{0} D\,\{\partial\eta/\partial t\}\,d\sigma = (1/D)\,\partial\left\{\int_{-1}^{0} D\,\eta\,d\sigma\right\}/\partial t = \partial[\eta]/\partial t \quad (A.2)$$

The integration of the second term yields

$$(1/D)\int_{-1}^{0} D\,\{\partial(uD)/\partial x\}\,d\sigma = \partial\left\{\int_{-1}^{0} u\,D\,d\sigma\right\}/\partial x = \partial[uD]/\partial x \quad (A.3)$$

The vertical integration of the third term of (A.1) is derived in the same fashion as (A.3). Due to the top and bottom boundary conditions, the last term of (A.1) vanishes.

Acknowledgments This study was supported by a short grant from UPM.

REFERENCES

1 Camerlengo A. and Demmler M. (1996) *Pertanika J. Sci. & Tech.*, **4(1)**: 111.

2 Kasahara A., (1974) *Mon. Wea. Rev.* **102**: 509.

3 Simons T. J., (1974) *J. Phys. Oceanogr.* **4**: 507.

4 Blumberg A. and Mellor G., (1983) *J. Geophys. Res.* **88**: 4579.

5 Madala R. V. & Piacsek S. A., (1977) *J. Comp. Phys.* **23**: 167-178.